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Electron Scattering from Thermal Vibrations in a Liquid

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It is shown that the law of conservation of momentum in the electron-phonon system may not necessarily hold in the case of scattering of electrons by the thermal vibrations in a liquid, as a consequence of the absence of strict periodicity in a liquid conductor. Therefore, in addition to the usual scattering of electrons by the phonons in a liquid, there exists an additional phonon-liquid scattering. The electron mean free path length corresponding to this scattering in liquid metals and semiconductors has been computed.

I. INTRODUCTION

QUANTUM-mechanical investigation shows[1] that the behavior of electrons in a liquid is similar to the behavior of electrons in a crystal; in particular, the electrons have a band energy spectrum and move as quasi-free particles. The scattering of electrons as a consequence of the disruption of long range order in liquids was calculated in Ref. 2. But, in addition to this scattering, which is specific to the liquid, there takes place (just as in crystals) a scattering of the electrons by the thermal vibrations which can play a dominant role at a high temperature of fusion. In the present research, we consider the scattering of electrons in the liquid by the thermal vibrations.

It was shown in Ref. 2 that the quasi-Bloch functions of the form

$$\psi_{\mathbf{k}} = G^{-1/2} u_{\mathbf{k}}(\xi)\, e^{i\mathbf{k}\mathbf{r}}, \qquad (1)$$

are an excellent zeroth approximation for the description of the electrons in the liquid. Here **k** is the wave vector of the electron and $u_{\mathbf{k}}(\xi)$ is a function which is almost periodic in the deformed coordinate system ξ in which the self-consistent potential for the electron in the liquid is periodic (see Ref. 1).

Thermal vibrations are considered in the theory of electrical conduction in the approximation of an elastic continuum, and the discreteness of the lattice is taken into account only for the determination of the limiting wave number of the phonons. It is natural that in this approximation there is no difference between crystal and liquid, so that we can use as the phonon, wave functions the same functions as in the theory of solids. Knowing the unperturbed wave functions (1) for the electrons, and making use of the usual functions for the phonons, we can repeat all the calculations for the liquid that have been used for solid conductors.

If we consider the interaction between electrons and phonons as a small perturbation, then the eigenfunction of the unperturbed problem is the product of the electron function and a function of all the oscillators of the lattice, and we can expand the wave function of the perturbed problem in a series of these products:

$$\Psi(t) = \sum c(\mathbf{k}, N_{\mathbf{q}j}, t)\psi^0_{\mathbf{k}}(r)\prod_{\mathbf{q},j}\psi_{N_{\mathbf{q}j}}(a_{\mathbf{q}j}) \tag{2}$$

$$\times\exp\left\{-\frac{it}{\hbar}\left[E_{\mathbf{k}} + \sum_{\mathbf{q}}\sum_{j}\left(N_{\mathbf{q}j} + \frac{1}{2}\right)\hbar\omega_{\mathbf{q}j}\right]\right\}.$$

As is known from Eq. (34.7) of Ref. 3, the expansion coefficients of c depend on the time and, for small t, satisfy the equation

$$i\hbar\dot{c}(\mathbf{k}', N'_{\mathbf{q}j}) = \int\psi^*(\mathbf{k}', N_{\mathbf{q}j})\,U\psi(\mathbf{k}, N_{\mathbf{q}j})\,d\tau \tag{3}$$

$$\times\exp\left\{\frac{it}{\hbar}\left[E_{\mathbf{k}'} - E_{\mathbf{k}} + \sum_{\mathbf{q}}\sum_{j}(N'_{\mathbf{q}j} - N_{\mathbf{q}j})\,\hbar\omega_{\mathbf{q}j}\right]\right\}.$$

The symbols here are standard: $\mathbf{k}$ and $\mathbf{k}'$ are the wave vectors of the electron; $\mathbf{q}$ is the wave vector of the phonon; j is the direction of its polarization; $N_{\mathbf{q}j}$ is the number of phonons with wave vector $\mathbf{q}$ and polarization direction j; $E_{\mathbf{k}}$ and $E_{\mathbf{k}'}$ are the energies of the electron for the corresponding index values of the wave vector, $\omega_{\mathbf{q}j}$ is the frequency of a phonon with given $\mathbf{q}$ and j. Integration over $d\tau$ consists of integration both over the radius vector of the electron, $\mathbf{r}$, and over the oscillator coordinates $a_{\mathbf{q}j}$.

The perturbation energy U has a different expression in the cases of acoustical and optical thermal vibrations. Initially, we shall consider the scattering by acoustical vibrations---the only vibrations in metals and atomic semiconductors.

2. MEASUREMENT OF THE ELECTRON DISTRIBUTION FUNCTION UNDER THE ACTION OF THERMAL VIBRATIONS

In the deformed-ions approximation the perturbation energy is equal to

$$U = -\mathbf{u}\,\mathrm{grad}\,V, \tag{4}$$

where V is the self-consistent potential for the electron and $\mathbf{u}$ is the displacement of the ions or atoms in the thermal vibrations:

$$\mathbf{u} = G^{-3/2}\sum_{\mathbf{q}}\sum_{j=1}^{3}\mathbf{e}_{\mathbf{q}j}(a_{\mathbf{q}j}e^{i\mathbf{q}\mathbf{r}} + a^*_{\mathbf{q}j}e^{-i\mathbf{q}\mathbf{r}}), \tag{5}$$

G^3 is the number of atoms in the elementary cell, $\mathbf{e}_{\mathbf{q}j}$ is the state of polarization of the phonons.

The integral over the electron coordinates in Eq. (3) can, after substitution of Eq. (1) for $\psi_{\mathbf{k}}(r)$ and $\psi_{\mathbf{k}'}(r)$, be represented in the form of a sum of integrals over the G^3 deformed elementary cells:

$$\mathbf{K}^+ = G^{-3}\sum_{\mathbf{n}}e^{i(\mathbf{k}+\mathbf{q}-\mathbf{k}')\mathbf{n}}\int e^{i(\mathbf{k}+\mathbf{q}-\mathbf{k}')\mathbf{r}'} \tag{6}$$

$$\times\,\mathrm{grad}\,V(\mathbf{r}')\,u_{\mathbf{k}}(\mathbf{r}')\,u^*_{\mathbf{k}'}(\mathbf{r}')\,d\tau_0,$$

where $\mathbf{n}$ is the radius vector of the n th site, $\mathbf{r}' = \mathbf{r} - \mathbf{n}$.

In a crystal, as a consequence of the periodicity of $V(r)$ and $u_{\mathbf{k}}(\mathbf{r})$, the integral over the elementary cell does not depend on the number of cells, but the values of the vector $\mathbf{n}$ form a regular lattice in space. Therefore, the sum over $\mathbf{n}$ differs from zero only if

$$\mathbf{k} \pm \mathbf{q} = \mathbf{k}'; \tag{7}$$

the sign (+) refers to K^+ and the sign (−) to K^-. Physically, the condition (7) signifies the law of conservation of momentum in the electron-phonon system, since K^+ describes the absorption of a phonon and K^- the emission of one. Under the condition (7), summation over $\mathbf{n}$ simply yields the factor G^3.

As noted in Ref. 2, $u_{\mathbf{k}}$ for a liquid, in the isotropic approximation, is a periodic function of the deformed coordinates ξ; therefore, when condition (7) is satisfied, the integral over the deformed elementary cell does not depend on the number of cells, with accuracy up to a small quantity of order ϵ---the degree of disruption of local order, i.e., the relative deformation of the elementary cell. There is no point to keeping a correction of order ϵ, since the perturbation itself is taken to be a small quantity, and these corrections would be of second order of smallness. Thus, when condition (7) is satisfied, the situation in the liquid is no different from that in the crystal.

It is easy to see that all subsequent calculations carried out for crystals will be valid with accuracy to corrections of order ϵ and the same results will be given for liquids. This is purely phonon scattering of the electrons, which is entirely independent of the disruption of long range order in the liquid.

However, if Eq. (7) is not observed, then it is not possible, in the case of a liquid, to prove that the sum over $\mathbf{n}$ [in Eq. (6)] vanishes. Actually, in the first place, in a liquid, the distant elementary cells are oriented at any angle relative to one another; therefore, the corresponding vectors $\mathbf{r}'$ in the different cells make different angles with the constant vector **k** ± **q** − **k**′, i.e., the exponential factors under the integral sign, and the integrals over the elementary cells themselves, depend on the number of cells. Second, the vectors **n** in the liquid no longer form a regular lattice, but are random quantities.

We show that the summation over the G^3 deformed elementary cells gives the factor $G^3\epsilon^2$. In the isotropic approximation, the integral over the elementary cell in Eq. (6) can be taken from under the summation sign; the problem then reduces to the calculation of the sum

$$\sum = \sum_{\mathbf{n}} e^{i\mathbf{q}'\mathbf{n}}, \tag{8}$$

where for brevity we put **k** ± **q** − **k**′ = **q**′. The vectors **n** in the liquid are random quantities but are not completely independent of one another, since the differences of the vectors **n**, relating to neighboring cells, differ in modulus from the lattice constant only by corrections of order ϵ.

As a preliminary we calculate the analogous sum for a one-dimensional liquid model:

$$\sum\nolimits_1 = \sum_{n=1}^{G} e^{iqx_n}, \tag{9}$$

where

$$qa = 2k\pi/G; \quad x_n = x_{n-1} + a(1 + \varepsilon\gamma_n), \tag{10}$$

a = lattice constant; $k = 0, 1, 2, \ldots$; G is the number of atoms in the chain; $\epsilon \ll 1$; γ_n are random variables (see Ref. 1). The computation of the sum is similar to the calculation of the length of a polymer molecule with restricted turning of the links (see, for example, Ref. 4, pp. 409-411). We therefore make use of the similar claculations there. We determine the square of the modulus of the sum:

$$\left|\sum\nolimits_1\right|^2 = \sum_{n=1}^{G} e^{iqx_n} \cdot \sum_{m=1}^{G} e^{-iqx_m} \tag{11}$$

$$= G + \sum_{n<m}\sum [e^{iq(x_n - x_m)} + e^{iq(x_m - x_n)}].$$

It follows from (10) that

$$x_n = na + a\varepsilon \sum_{l=1}^{n} \gamma_l; \quad x_m = ma + a\varepsilon \sum_{l=1}^{m} \gamma_l. \tag{12}$$

Substituting (12) in (11), we transform this expression to the form

$$\left|\sum\nolimits_1\right|^2 = G + \sum_{n<m}\sum \left\{ e^{iqa(n-m)} \left[\cos\left(\varepsilon aq \sum_{l=n+1}^{m} \gamma_l\right) - i\sin\left(\varepsilon aq \sum_{l=n+1}^{m} \gamma_l\right)\right] + e^{iqa(m-n)} \left[\cos\left(\varepsilon aq \sum_{l=n+1}^{m} \gamma_l\right) + i\sin\left(\varepsilon aq \sum_{l=n+1}^{m} \gamma_l\right)\right]\right\}. \tag{13}$$

Transforming to mean values, we note that inasmuch as the y_l take on positive and negative values with equal probability, the sines vanish and the average values of the cosine sum will be equal to the products of the mean cosine components. Since the mean square values of all the y_l are the same, the product reduces to the order of the quantity $\overline{\cos(\epsilon aq y_l)}$. Consequently,

$$\overline{\left|\sum\nolimits_1'\right|^2} = G + \sum_{n<m}\sum \overline{[\cos(\varepsilon aq\gamma_l]}^{m-n} [e^{iqa(m-n)} + e^{iqa(n-m)}]. \tag{14}$$

The summation is carried out without difficulty and yields

$$\overline{\left|\sum\nolimits_1'\right|^2} = G + \frac{G\zeta\eta}{1 - \zeta\eta} + \frac{G\zeta/\eta}{1 - \zeta/\eta} - \zeta\eta\frac{(1 - \zeta^G\eta^G)}{(1 - \zeta\eta)^2} - \frac{\zeta}{\eta}\frac{(1 - \zeta^G/\eta^G)}{(1 - \zeta/\eta)^2}, \tag{15}$$

where, for brevity, we have put

$$\eta = e^{iqa} = e^{2\pi ik/G}; \quad \zeta = \overline{\cos(\varepsilon aq\gamma_l)}. \tag{16}$$

Discarding the last two terms in Eq. (15), which do not contain the large multiplicative factor G, we get

$$\overline{\left|\sum\nolimits_1\right|^2} \approx G\frac{1 - \zeta^2}{1 - \zeta\eta - (\zeta/\eta) + \zeta^2}. \tag{17}$$

Expanding Eq. (16) in a series, and keeping in mind the smallness of ϵ, and that $\overline{\gamma_l^2} = 1$, we get from Eq. (17), for $q \neq 0$,

$$\overline{\left|\sum\nolimits_1\right|^2} \approx G\varepsilon^2. \tag{18}$$

For $q = 0$, the approximate equation (17) is not suitable, and the more accurate expression (15) yields $|\Sigma_1|^2 = G^2$, in correspondence with the usual theory.

The three-dimensional case can be reduced to one-dimensional if we direct the x axis along the vector q^1 and the fundamental region of G^3 atoms into G^2 chains of atoms, parallel to the x axis, with G atoms in each. Then we can accomplish summation along the chain as in the one-dimensional case; we get the factor $G\epsilon^2$ in the expression for the mean square modulus of the sum. The summation over the G^2 chains also yields a factor G^2; hence, the sum reduces to multiplication by $G^3\epsilon^2$.

Consequently, scattering of electrons by phonons is possible in a liquid without satisfying the law of conservation of momentum inside the electron-phonon system. This can explain the fact that the lattice, as a consequence of the disruption of long range order in it, can take on additional momentum in the emission or absorption of a phonon by an electron. A similar effect takes place in the case of impurities in crystals.

We call this additional scattering of the electrons "phonon-liquid," since both thermal vibrations and disruption of long range order are essential here in equal degree. At first glance, consideration of phonon-liquid scattering can be regarded as inconsequential, inasmuch as we neglect corrections of order ϵ for the usual phonon scattering in a liquid. However, there were only small corrections to the numerical values of the coefficients in such a case, whereas phonon-liquid scattering is a new physical phenomenon with specific properties and some dependence on the energy of the electron, and therefore (in certain cases) it can play a role in spite of the smallness of ϵ. Let us consider the mean free path length l_{pl} connected with phonon-liquid scattering.

Because of the exponential factor in the integral over the elementary cell, this integral, strictly speaking, depends on $\mathbf{k} + \mathbf{q} - \mathbf{k}'$. However, the factor grad V has a maximum value inside the atomic core, where $\mathbf{r}'$ is small; therefore, the effect of the exponential factor is insignificant (see Ref. 3, p. 227), and we neglect it. Consideration of the exponential factor would only have given rise to a coefficient close to unity.

The summation of G^3 randomly sign-changing quantities gives a factor G^3 in the expression for the square of the modulus of the sum. Integration is carried out over the oscillator coordinates, as in the case of a metal. Integrating Eq. (3) with respect to time, we get, after some transformations (see Ref. 3, pp. 187-190):

$$|c(\mathbf{k}', N_{\mathbf{q}j} - 1, t)|^2 = (2G^{-6}C^2\varepsilon^2/9M\hbar)(\mathbf{k} - \mathbf{k}') \times \mathbf{e}_{\mathbf{q}j} N_{\mathbf{q}j} \Omega(E_{\mathbf{k}'} - E_{\mathbf{k}} - \hbar\omega_{\mathbf{q}j}), \tag{19}$$

$$|c(\mathbf{k}', N_{\mathbf{q}j} + 1, t)|^2 = (2G^{-6}C^2\varepsilon^2/9M\hbar) \times (\mathbf{k} - \mathbf{k}')\mathbf{e}_{\mathbf{q}j}(N_{\mathbf{q}j} + 1)\Omega(E_{\mathbf{k}'} - E_{\mathbf{k}} + \hbar\omega_{\mathbf{q}j}). \tag{20}$$

$$C = \frac{\hbar^2}{2m}\int |\operatorname{grad} u|^2 d\tau_0; \quad \Omega(x) = 2\,\frac{1 - \cos(xt/\hbar)}{(x/\hbar)^2}.$$

Inasmuch as $\mathbf{k} - \mathbf{k}' \neq \pm\mathbf{q}$, the electrons, in contrast to ordinary scattering, interact not only with longitudinal, but also with transverse, waves. Because of the δ-character of the function $\Omega(x)$, the law of energy conservation is satisfied in the electron-photon system. This means that the disrupted lattice, taking on additional momentum, does not absorb additional energy for the photon.

In order to obtain a change in the electron distribution function at the expense of scattering, we must multiply Eq. (19) by the probability for the existence of the initial state, and also by the probability of non-occupation of the final state, and sum over all initial values of the wave vector $\mathbf{k}$ of the electron, over all values of the wave vector of the photon $\mathbf{q}$, and over the directions of its polarization j. We emphasize that, in contrast to the ordinary case of phonon scattering, the summation here over $\mathbf{k}$ is taken independently of the summation over $\mathbf{q}$. As a result, we obtain:

$$\begin{aligned} f(\mathbf{k}', t) - f(\mathbf{k}', 0) = {} & \frac{2C^2G^{-6}\varepsilon^2}{9M\hbar} \sum_{\mathbf{k},\,\mathbf{q},\,j} [(\mathbf{k} - \mathbf{k}')\mathbf{e}_{\mathbf{q}j}]^2 \\ & \times \{\Omega(E_{\mathbf{k}'} - E_{\mathbf{k}} + \hbar\omega_{\mathbf{q}j})[f(\mathbf{k})(1 - f(\mathbf{k}')) \\ & \times (N_{\mathbf{q}j} + 1) - f(\mathbf{k}')(1 - f(\mathbf{k}))N_{\mathbf{q}j}] \\ & + \Omega(E_{\mathbf{k}'} - E_{\mathbf{k}} - \hbar\omega_{\mathbf{q}j})[f(\mathbf{k})(1 - f(\mathbf{k}'))N_{\mathbf{q}j} \\ & - f(\mathbf{k}')(1 - f(\mathbf{k}))(N_{\mathbf{q}j} + 1)]\}. \end{aligned} \tag{21}$$

3. THE MEAN FREE PATH OF THE ELECTRON AS DETERMINED BY PHONON-LIQUID SCATTERING

We introduce the spherical coordinate system k, θ, Φ in the space $\mathbf{k}$, with axis k_x and the coordinates q, ϑ, φ in the space $\mathbf{q}$, with axis along $\mathbf{k}-\mathbf{k}'$, and we replace the summations over k and q by integration, with the help of the formulas

$$\sum_k F(k) = \Omega_0 (G/2\pi)^3 \int k^2\,dk \sin\theta\, d\theta\, d\Phi F(k), \tag{22}$$

$$\sum_q F(q) = \Omega_0 (G/2\pi)^3 \int q^2 dq \sin\vartheta\, d\vartheta\, d\varphi F(q),$$

Ω_0 is the volume of the elementary cell. We shall consider that the phonons are in thermal equilibrium, while the electrons have small deviations from thermal equilibrium at the expense of the electric field, or the temperature gradient, which act along the x direction. In this case,

$$N_{\mathbf{q}j} = \frac{1}{\exp\{\hbar\omega_{qj}/kT\}-1}; \tag{23}$$

$$f = f_0 - k_x \frac{\partial f_0}{\partial E}\chi(E);$$

$$f_0 = [\exp\{(E-\mu)/kT\}+1]^{-1},$$

$\chi(E)$ is a small correction whose square we shall neglect, μ is the Fermi potential. We shall consider the energy to be dependent only on the modulus of k or k'.

Integration with respect to φ yields the factor 2π, ϑ enters only in the scalar product $(\mathbf{k}-\mathbf{k}')\mathbf{e}_{qj}$. Let $j = 1$ be the longitudinal wave and $j = 2,3$ the transverse, where $\mathbf{e}_{q2}$ lies in the plane $\mathbf{k}-\mathbf{k}'$, $\mathbf{q}$, and $\mathbf{e}_{q3}$ are perpendicular to it. Then we easily obtain

$$\int [(\mathbf{k}-\mathbf{k}')\mathbf{e}_{q1}]^2 \sin\vartheta\, d\vartheta = \frac{2}{3}(\mathbf{k}-\mathbf{k}')^2; \tag{24}$$

$$\int [(\mathbf{k}-\mathbf{k}')\mathbf{e}_{q2}]^2 \sin\vartheta\, d\vartheta = \frac{4}{3}(\mathbf{k}-\mathbf{k}')^2.$$

In the integration over q, we take into account the δ-character of the function $\Omega(x)$. All the factors save Ω can be removed from the integrand, setting

$$\omega_{\mathbf{q}j} = \pm (E_k - E_{k'})/\hbar; \tag{25}$$

$$q = \omega_{\mathbf{q}j}/u_j = \pm(E_k - E_{k'})/\hbar u_j,$$

where u_1 and u_2 are the velocities of the longitudinal and transverse waves, respectively. The remaining integral gives

$$\int_0^\infty \Omega(E_{k'} - E_k - \hbar\omega_{qj})\,dq \approx 2\pi t/u_j. \tag{26}$$

In the substitution of Eqs. (23) and (25) in (21), all the terms of zeroth order of smallness vanish, leaving only terms with $f_1(E_k)$ and $f_1(E_{k'})$. Further,

$$(\mathbf{k}-\mathbf{k}')^2 = k'^2 + k^2 - 2kk'\cos\vartheta', \tag{27}$$

where ϑ' is the angle between $\mathbf{k}$ and $\mathbf{k}'$. If we denote by Φ the angle from the plane $\mathbf{k}'$, x, then

$$\cos\vartheta' = \cos\theta\cos\theta' + \sin\theta\sin\theta'\sin\Phi. \tag{28}$$

The term in $\sin\Phi$ vanishes upon integration, and all the remaining terms are multiplied by 2π.

The limits of the integration over k are obtained from the condition

$$E(k^j_{\max}) = E_{k'} + \hbar\omega_j,\ E(k^j_{\min}) = \begin{cases} E_{k'} - \hbar\omega_j \\ 0 \end{cases} \text{for}\quad E_{k'} \gtrless \hbar\omega_j; \tag{29}$$

ω_1 and ω_2 are the limiting frequencies of the longitudinal and transverse waves.

Carrying out the integration over θ, substituting the value of $(\partial f/\partial s)_{\mathrm{pl}}$ in the kinetic equation for the case in which the electric field E_x acts along the x axis, and setting $k'\cos\theta' = k'_x$, we get the Bloch integral equation for the function $\chi(E)$, which, after several transformations (which make use of the explicit functions for N and f_0), takes the form

$$\begin{aligned} \sum_{j=1}^{2} \frac{j}{u_j^3} \Bigg\{ \int_{k'}^{k^j_{\max}} & N_{qj} k^2 dk\,(E_k - E_{k'}) \\ & \times \exp\left\{\frac{E_k - E_{k'}}{kT}\right\} \frac{f_0(E_k)}{f_0(E_{k'})} \Big[\frac{2}{3}k^2\chi(E_k) \\ & + (k^2 + k'^2)\chi(E_{k'})\Big] dk \\ + \int_{k^j_{\min}}^{k'} & N_{qj} k^2 dk\,(E_{k'} - E_k) \frac{f_0(E_k)}{f_0(E_{k'})} \Big[\frac{2}{3}k^2\chi(E_k) \\ & + (k^2 + k'^2)\chi(E_{k'})\Big]\Bigg\} dk = \frac{27M\pi'\hbar eE_x}{k'C^2\Omega_0^2\varepsilon^2}\,\frac{dE}{dk'}. \end{aligned} \tag{30}$$

Generalization to the case in which there is a temperature or concentration gradient in addition to the electric field presents no additional difficulties.

For computation of the integral in Eq. (30), it is appropriate to introduce the notation

$$\frac{|E_k - E_{k'}|}{kT} = y, \quad \frac{E_{k'}}{kT} = \eta, \quad \frac{E_{k'} - \mu}{kT} = \zeta, \quad \frac{\hbar\omega_j}{kT} = y_j, \qquad (31)$$

$$k' = k(\eta), \quad k = k(\eta \pm y),$$

and transform to integration over y. The left-hand side of Eq. (30) takes the form

$$\frac{(kT)^2 k'^4 \chi(E_{k'})}{dE_{k'}/dk'} \sum_{j=1}^{2} \frac{j}{u_j^3} (I_j' + I_j''), \qquad (32)$$

$$I_j' = \int_0^{y_j} \frac{ye^y}{e^y - 1} \, \frac{e^\zeta + 1}{e^{\zeta+y} + 1} \left[\frac{2}{3} \frac{k^2(\eta + y)}{k^2(\eta)} \frac{\chi(\eta + y)}{\chi(\eta)} + \frac{k^2(\eta + y)}{k^2(\eta)} + 1 \right] \frac{dk(\eta + y)}{dk(\eta)} \, dy. \qquad (33)$$

$$I_j'' = \int_0^{y^*} \frac{y}{e^y - 1} \, \frac{e^\zeta + 1}{e^{\zeta-y} + 1} \left[\frac{2}{3} \frac{k^2(\eta + y)}{k^2(\eta)} \frac{\chi(\eta + y)}{\chi(\eta)} + \frac{k^2(\eta + y)}{k^2(\eta)} + 1 \right] \frac{dk(\eta + y)}{dk(\eta)} \, dy,$$

where, in accord with Eq. (19), $y^* = y_j$ for $\eta > y_j$, $y^* = \eta$ for $\eta < y_j$.

The expressions in front of the square brackets in Eq. (33) have a maximum at $y = 0$ and decrease rapidly thereafter. Therefore, we make no great error if we take the values at $y = 0$ for the slowly varying functions $k^2(\eta + y)$, $\chi(\eta + y)$ and $dk(\eta + y)$. This certainly is the case for metals and for semiconductors at high temperatures, since in both cases $\eta \gg y$ for the vast majority of the electrons and these functions vary within relatively narrow limits.

These assumptions are somewhat less valid for semiconductors at low temperatures, but for this case, as we whall see below, phonon-liquid scattering does not play an important role, and a rough estimate suffices.

We note that the assumption $\chi = \text{const}$ also applies in the theory of electrical conductivity in the region of very high temperatures. In our case, the structure of the Bloch integral equation is much more favorable for the assumption mentioned; therefore, it can be extended to any temperature. This means that the mean free path of the electron, connected with the phonon-liquid scattering, exists at any temperature.

As a result of the assumptions we have made, the expressions for I_j' and I_j'' are greatly simplified:

$$I_j' = \frac{8}{3} \int_0^{y_j} \frac{ye^y}{e^y - 1} \, \frac{e^\zeta + 1}{e^{\zeta+y} + 1} \, dy; \qquad (34)$$

$$I_j'' = \frac{8}{3} \int_0^{y^*} \frac{y}{e^y - 1} \, \frac{e^\zeta + 1}{e^{\zeta+y} + 1} \, dy.$$

Substituting Eq. (22) in Eq. (20), solving this equation for $\chi(E_{k'})$ and setting $\Omega_0 = a^3$, we obtain the free path length of the electron as determined by the phonon-liquid scattering:

$$l_{\text{pl}} = k'\chi(E_{k'})/eE \qquad (35)$$

$$= 27M\pi^3\hbar / k'^4 (kT)^2 C^2 \varepsilon^2 a^6 \sum_{j=1,2} j u_j^{-3} (I_j' + I_j'').$$

I_j' is a function of y_j, I_j'' for $\epsilon > y_j$ is also a function of y_j, and for $\epsilon < y_j$ it is a function of ϵ. From Eq. (21), $y_1 = \Theta/T$, where Θ is the Debye temperature. If we assume that for longitudinal and transverse waves there exists just one limiting value of the wave vector q_0, then, evidently, $y_2 = \Theta u_2 / T u_1$; consequently, I_j' and I_j'' are functions of Θ/T. Explicit forms of this function can be obtained in the region of high temperatures ($T \gg \Theta$) and in the region of low temperatures ($T \ll \Theta$).

For $T \gg \Theta$, $y \ll 1$, for most electrons, $y_j < \epsilon$, which means $y^* = y_j$,

$$I_1' = I_1'' = 8\Theta/3T, \quad I_2' = I_2'' = 8\Theta u_2/3Tu_1. \qquad (36)$$

Substituting Eq. (36) in Eq. (35), and using the relations

$$u_1 = k\Theta/\hbar q_0, \quad q_0 = (2\pi/a)(3/4\pi)^{1/3}, \qquad (37)$$

we get, after several transformations,

$$l_{\text{pl}} = (9\pi Mak\Theta / 32\beta C^2 \varepsilon^2 \hbar^2 (ak')^4) \times (\Theta/T)(dE_{k'}/dk')^2. \qquad (38)$$

Here β is a number close to unity:

$$\beta = 1/3 + 2u_1^2/3u_2^2. \qquad (39)$$

For $T \ll \Theta$ we investigate metals and semiconductors separately. For metals, the electrons in a narrow band around the Fermi level play the important role. Therefore, we set $\zeta = 0$. Further, $\eta > y_j$, $y^* = y_j$, and since $y_j \gg 1$, then the upper limit of integration in both integrals of (34) can be extended to infinity. Then

$$I'_j = I''_j = 2\pi^3/3 \tag{40}$$

(see Ref. 5, pp. 134, 350); I'_j and I''_j do not depend on j. Therefore,

$$l_{\mathrm{pl}} = (9Mak\Theta / 8\pi\beta' C^2\varepsilon^2\hbar^2 (ak')^4) \times (\Theta / T)^2 (dE_{k'} / dk')^2, \tag{41}$$

$$\beta' = {}^1/_3 + 2u_1^3 / 3u_2^3. \tag{42}$$

For semiconductors, $\zeta \gg 1$, and we can neglect unity in comparison with e^{ζ}. In the integral for I'_j, we can again extend the integral to infinity. Then

$$I'_j = 4\pi^2/9 \tag{43}$$

(see Ref. 5, pp. 134, 244).

For most electrons in the conduction band of a semiconductor, η has the order of unity for $T \ll \Theta$, $\eta < y$, which means that $y^* = \eta$ and it is easy to see that

$$\text{for } \eta < \pi^2/6 \quad I'_j \leqslant 8\eta/3; \qquad \text{for } \eta > \pi^2/6 \quad I''_j \leqslant 4\pi^2/9, \tag{44}$$

the equality holds: in the first case for $\eta \ll 1$, in the second case for $\eta \gg 1$.

Consequently,

$$l_{\mathrm{pl}} = (27Mak\Theta\alpha(E_{k'}) / 16\pi\beta' C^2\varepsilon^2\hbar^2 (ak')^4) \times (\Theta / T)^2 (dE_{k'} / dk')^2. \tag{45}$$

As $E_{k'}$ increases from zero to infinity, the coefficient $\alpha(E_{k'})$ changes from 1 to ½.

It is of interest to compare the mean free paths obtained here with the value l_{p} which is defined by the usual phonon scattering. In accord with Ref. 3, Eqs. (36.8), (37.15) and (45.16) for metals:

for $T \gg \Theta$

$$l_{\mathrm{p}} = (Mk'^2a^3k\Theta / \pi^3\hbar^2C^2)(\Theta / T)(dE_{k'} / dk')^2, \tag{46}$$

for $T \ll \Theta$

$$l_{\mathrm{p}} = (Mk'^2a^3k\Theta / 4\pi^3 124{,}4\hbar^2C^2) \times (\Theta / T)^5 (dE_{k'} / dk')^2. \tag{47}$$

For semiconductors at arbitrary temperature

$$l_{\mathrm{p}} = (9(4\pi/3)^{2/3}Mak\Theta / 16\pi\hbar^2C^2(ak')^2) \times (\Theta / T)(dE_{k'} / dk')^2. \tag{48}$$

Consequently, for metals at $T \gg \Theta$,

$$l_{\mathrm{pl}} / l_{\mathrm{p}} \approx 9\pi / (ak')^6\varepsilon^2, \tag{49}$$

and for $T \ll \Theta$,

$$\frac{l_{\mathrm{pl}}}{l_{\mathrm{p}}} \approx \frac{5600}{(ak')^6\varepsilon^2}\left(\frac{T}{\Theta}\right)^3. \tag{50}$$

For semiconductors, for $T \gg \Theta$,

$$l_{\mathrm{pl}} / l_{\mathrm{p}} \approx 2 / (ak')^2\varepsilon^2, \tag{51}$$

and for $T \ll \Theta$,

$$l_{\mathrm{pl}} / l_{\mathrm{p}} \approx \Theta / (ak')^2\varepsilon^2 T. \tag{52}$$

In metals close to the filling limit, ak' is approximately equal to π. Therefore, in accord with Eqs. (39) and (40), for $\epsilon \geq 0.1$, and in the region of high and very low temperatures, l_{pl} can be the same order of magnitude as l_{p}; then the phonon-liquid scattering will play a significant role. In semiconductors, $ak' < 1$ for most of the electrons, and is the smaller, the lower the temperature. Therefore, in accord with Eqs. (41) and (42), $l_{\mathrm{pl}} \gg l_{\mathrm{p}}$. In the majority of cases, phonon-liquid scattering can be neglected in semiconductors, and only for high temperatures and comparatively high values of ϵ does it give a small correction to the ordinary thermal scattering.

4. CASE OF OPTICAL SCATTERING

In accord with Fröhlich[6], the perturbation energy in the interaction of the electrons with the optical vibrations of the lattice has the form

$$U = -(4\pi i e^2 / 2a^3 q)(2G^3)^{-1/2} \sum_{\mathbf{q}} (a_{\mathbf{q}} e^{i\mathbf{qr}} - a_{\mathbf{q}}^{*} e^{-i\mathbf{qr}}), \tag{53}$$

where only the longitudinal optical vibrations interact with the electrons, since dipole polarization arises only for the longitudinal wave.

Subsequent calculation is carried out similar to the case of acoustical vibration; because of lack of space, we will not carry it out and will only note the essential differences and write down the final formulas.

The peculiar features of the calculation with optical vibrations are the following:

1. In the integral over the electron coordinates (6), the factor grad $V(\mathbf{r}')$ is lacking.

2. In the integral over the elementary cell, which is given in Eq. (6), we take from under the integral sign the mean value of the product $u_{\mathbf{k}} u^{*}_{\mathbf{k}'}$, equal to a^{-3}, taking into account that in most of the elementary cells it is almost constant. This actually takes place in the case of metals (see Ref. 3, p. 79); in the case of ionized matter, such an assumption can be considered as a limiting case.

We compute the remaining integral over an elementary cell in the form of cube with edge a, in which $\mathbf{r}'$ is excluded from the center of the cube. We direct the axes ξ_α along the edge of the cube and for brevity write $\mathbf{k} \pm \mathbf{q} - \mathbf{k}' = \mathbf{q}'$; then,

$$a^{-3} \int e^{i\mathbf{q}'\mathbf{r}'} d\tau_0 = \prod_{\alpha=1}^{3} \frac{\sin (q'_\alpha a/2)}{(q'_\alpha a/2)} \approx 1 - \frac{a^2 q'^2}{24}. \tag{54}$$

The latter expression is obtained by a series expansion in sines and a neglect of powers of aq' above the second. In this case, and also in taking $u_{\mathbf{k}} u^{*}_{\mathbf{k}'}$ from under the integral sign, we have exaggerated the value of the term which is a correction to the unity term in (54).

3. In Eq. (21) there is lacking a summation over j and in place of the factor $(2C^2/9) \times [(\mathbf{k} - \mathbf{k}')\mathbf{e}_{\mathbf{q}j}]^2$ there is $(e^4/a^6\omega_0)(1 - a^2 q'^2/12)$, and in place of $\omega_{\mathbf{q}j}$ the frequency of the optical vibration ω_0, which is independent of $\mathbf{q}$:

$$q'^2 = (\mathbf{k} - \mathbf{k}' \pm \mathbf{q})^2 \tag{55}$$

$$= (\mathbf{k} - \mathbf{k}')^2 + q^2 \pm 2q\,|\mathbf{k} - \mathbf{k}'| \cos \vartheta.$$

4. In the integration over ϑ, the term with $\cos \vartheta$ disappears and the remaining terms are multiplied by 2. Integration with respect to q gives

$$\int_0^{q_0} \left(1 - \frac{a^2 q'^2}{12}\right) dq = q_0 - \frac{a^2 (k - k')^2 q_0}{12} - \frac{a^2 q_0^3}{36}. \tag{56}$$

In semiconductors and for most electrons in the conduction band, $ak \ll 1$; therefore, we can neglect the second term in (46). But in such a case, taking it into account that $q_0 = (2\pi/a)(3/4\pi)^{1/3}$, we get

$$\int_0^{q_0} \left(1 - \frac{a^2 q'^2}{12}\right) dq \approx \left(\frac{0{,}57 \cdot 2\pi}{a}\right)\left(\frac{3}{4\pi}\right)^{1/3}, \tag{57}$$

i.e., consideration of the factor $e^{i\mathbf{qr}}$ in the integral over the elementary cell leads only to the numerical coefficient 0.57. If the work is done more accurately, this coefficient will be close to 1. Integration over Φ gives the factor 2π. In the integration over θ, terms with $k_x = k \cos\theta$ disappear, and the terms with $k'_x = k' \cos\theta'$ are multiplied by 2. In integrating over k, we make use of the δ-character of the function Ω. All the factors save Ωk are removed from under the integral sign. We set $E_k = E_{k'} \pm \hbar\omega_0$, extend the limit of the integration to infinity and take into consideration that in the semiconductor,

$$E_k = h^2 k^2 / 2m^*;\ E_{k'} = h^2 k'^2 / 2m^*, \tag{58}$$

where m^* is the effective mass of the electron. The integration over k gives

$$\int_{-\infty}^{\infty} \Omega (E_{k'} - E_k \pm h\omega_0)\, k\,dk = \left(\frac{2\pi m^*}{\hbar}\right) t. \tag{59}$$

If we set

$$I = \sqrt{2m^*(E_{k'} + h\omega_0)}\,[N + f_0(E_{k'} + h\omega_0] \tag{60}$$

$$+ \sqrt{2m^*(E_{k'} - h\omega_0)}\,[1 + N - f_0(E_{k'} - \hbar\omega)],$$

then, as the result of all the calculations, the mean free path length of the electron as determined by the phonon-liquid scattering on thermal optical vibrations is equal to

$$l_{\mathrm{pl}} = -\frac{k' f_1}{eE_x (df_0/dE)} \tag{61}$$

$$= \frac{\pi^2 M \hbar^2 \omega_0 a}{0{,}57\,(3/4\pi)^{1/3} e^4 m^* I \varepsilon^2} \frac{dE_{k'}}{dk'}.$$

With the help of Eqs. (23), (58), (60) and (61), the length of the free path can be computed in closed form for arbitrary temperature but the quantity is a rather complicated function of the temperature T and the electron energy $E_{k'}$. The expression for l can be greatly simplified in the regions of high and low temperatures, which are separtaed at the Debye temperature

$$\Theta = \hbar\omega_0 / k. \qquad (62)$$

For $T \gg \Theta$, $N \approx kT/\hbar\omega_0$; for most electrons, $E_{k'} \gg \hbar\omega_0$, for semiconductors, $f_0 \ll N$:

$$l \approx (2kT / \hbar\omega_0)\,\hbar k'. \qquad (63)$$

Substituting (63) in (61), we get, after several transformations,

$$l_{\mathrm{pl}} = \frac{a\pi^2}{0{,}57\,(3/4\pi)^{1/3}\varepsilon^2}\,\frac{M}{m^*}\left(\frac{\hbar\omega_0}{e^2/a}\right)^2\frac{E}{kT}\,\frac{1}{(k'a)^2}. \qquad (64)$$

For $T \ll \Theta$,

$$N \approx \exp\{-\hbar\omega_0/kT\} \ll 1.$$

The second term in Eq. (60) is absent since the energy for most electrons is insufficient to emit a phonon. Consequently,

$$l \approx \exp\{-\hbar\omega_0/kT\}\sqrt{2m^*\hbar\omega_0} \qquad (65)$$

and we get, after some algebra,

$$l_{\mathrm{pl}} = \frac{2a\pi^2}{0{,}57\,(3/4\pi)^{1/3}\varepsilon^2}\,\frac{M}{m^*}\,\frac{\hbar\omega_0 E_{k'}}{(e^2/a)^2}\,\frac{1}{(k'a)^2} \qquad (66)$$

$$\times\ \exp\{\hbar\omega_0/kT\}\sqrt{E_{k'}/\hbar\omega_0}.$$

Let us compare the expression obtained for the mean free path with the expressions for the mean free path obtained in the usual phonon scattering. In accord with Davydov and Shmushkevich [Ref. 7, Eqs. (3.97) and (3.98)] we have, for ionic crystals with ionic charge $z = 1$:

for $T \gg \Theta$,

$$l_{\mathrm{p}} = \frac{a}{2\pi}\,\frac{M}{m^*}\left(\frac{\hbar\omega_0}{e^2/a}\right)^2\frac{E_{k'}}{kT}, \qquad (67)$$

and for $T \ll \Theta$,

$$l_{\mathrm{p}} = \frac{a}{2\pi}\,\frac{M}{m^*}\left(\frac{\hbar\omega_0}{e^2/a}\right)^2\exp\{\hbar\omega_0/kT\}\sqrt{\frac{E_{k'}}{\hbar\omega_0}}. \qquad (68)$$

Consequently, for $T \gg \Theta$,

$$l_{\mathrm{pl}} / l_{\mathrm{p}} \approx 180\,(k'a\varepsilon)^{-2} \gg 1, \qquad (69)$$

and for $G \ll \Theta$,

$$l_{\mathrm{pl}} / l_{\mathrm{p}} \approx 360\,(k'a\varepsilon)^{-2}E_{k'}/\hbar\omega_0 \gg 1. \qquad (70)$$

Thus the investigation that we have carried out shows that, with accuracy up to small corrections, the ordinary thermal scattering of the electrons in a liquid can be considered according to the same formulas as in a solid. However, in liquids there also exists an additional phonon-liquid scattering, which can play an important role in liquid metals, while in semiconductors it is only a small correction or is negligibly small.

In conclusion, I take this opportunity to express my gratitude to L. E. Gurevich for his valuable discussions.

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Electron Detachment from Negative Halogen Ions in Collisions with Inert-Gas Atoms and Hydrogen Molecules

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Measurements have been made of the effective cross sections for electron detachment in the negative ions F^-, Cl^-, Br^- and I^- in collisions with atomic He, Ne, A, Kr and Xe in the ion-energy region from 200 to 2,000 ev. The cross sections have also been measured for collisions between Cl^-, Br^- and I^- ions and H_2 molecules. The following energy thresholds for electron detachment were found: Br^-, He (150 ev); Br^-, H_2 (160 ev); I^-, H_2 (320 ev); the threshold for I^-, He (280 ev) was established accurately. The results are interpreted in terms of energy transitions in a system consisting of two slowly approaching atomic particles.

1. INTRODUCTION

IN collisions of negative ions with atoms or molecules of a gas the excess electron associated with the negative ion can be detached, making a transition to an unbound state. It has been shown experimentally[1-3] that for ion energies of about 1,000 ev the effective cross sections for this process (which in the following we shall call collision-stripping of negative ions) may reach values of the order of 10^{-15} to 10^{-16} cm^2.

Negative-ion stripping has not been studied adequately although it is of more than usual interest, being a particularly simple process and one which makes easily available for examination inelastic collisions between atomic particles. A study of negative-ion stripping may, in general, prove useful in establishing general features of collisions between slow-moving atomic particles and, in particular in clarifying the question of energy thresholds in the excitation and ionization processes which occur in such collisions. It was our purpose in the present work to study negative-ion stripping in a more detailed and systematic fashion than was done in Refs. 1 and 2, and as far as possible, to remove certain experimental deficiencies in this earlier work.

It was decided to investigate the stripping process in negative halogen ions since these form a homogeneous group with similar structure in the outer part of the electron shell (a closed group of p-electrons). Atoms of the inert gases were chosen as the target particles since these also have closed electron shells. We were also interested in extending the data on energy thresholds for the stripping effect. These thresholds are most accessible to measurement in collisions of heavy negative ions with light atoms[4]. Hence, careful measurements of the ion-stripping effect were made for Cl^-, Br^- and I^- in helium, and these were supplemented by studies in hydrogen.

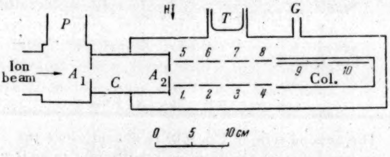

FIG. 1. Diagram of the collision chamber.

The ion-stripping was studied for ion energies ranging from 200 to 2,000 ev; in this energy region the stripping effect is not complicated by the presence of the other inelastic processes which occur at higher ion energies[5].

2. EXPERIMENTAL ARRANGEMENT

To observe negative-ion stripping, use was made of a scheme in which the slow electrons produced in the passage of a beam of negative ions through a gas-filled chamber are detected. The experimental arrangement used in the present work is similar to that described in Ref. 1.

The source of negative halogen ions was a hot-cathode gaseous discharge in vapors of one of the salts KF, NaCl, NaBr or NaI. A homogeneous beam of halogen ions was extracted by a magnetic mass-analyzer with a revolving power of 40 and, following additional collimation, introduced into the collision chamber (Fig. 1). Here C is the collimator (the diameter of aperture A_1 is 4.0 mm, that of aperture A_2 is 2.0 mm). Electrons detached from negative ions in collisions with gas atoms were collected by two pair of semi-cylindrical electrodes, *2* and *6* and *3* and *7*, which surrounded the ion beam. The measuring electrodes were located between two pair of guard electrodes, *1* and *5* and *4* and *8*. The collector Col for the primary ion beam was a deep cylinder designed to prevent the escape of secondary electrons. For this same reason there were two retarding electrodes, *9* and *10*, inside the collector; these provided an electric field which was used to deflect secondary electrons. A large liquid-nitrogen trap *T* was used to freeze out condensable vapors in the collision chamber. The gas was admitted to the chamber through the tube *G* and was pumped out through the port at *P*. There was a pressure differential across the aperture A_2. With gas pressure in the collision chamber of a few times 10^{-4} mm Hg the pressure at the port *P* was approximately 100 times smaller.

The collision chamber was filled with so-called spectrally pure inert gases. Before being admitted to the chamber the gas passed through a coil which was cooled by liquid nitrogen (in the case of Xe it was cooled by solid CO_2 in acetone). The hydrogen was purified by passage through a heated palladium thimble. The gas purity was checked by mass-spectroscopic analysis. The only spurious line in the mass-spectrographic measurements was found to be one due to H_2O^+, the intensity of which was less than 1 percent of the intensity of the test gas. Aside from this no other contamination of the inert gas was found.

The current in the primary ion beam was measured by a dc electrometer amplifier connected to the collector Col. A second electrometer amplifier (type EMA-1) with a maximum usable sensitivity of 1×10^{-15} amp/div was used to measure the current due to electrons which appeared in the collision chamber as a result of ion stripping.

A particular effort was made to reduce the effect of the fringing magnetic field *H* of the mass-analyzer on the electron collection in the collision chamber. Because of the proximity of the ion source to the magnet of the mass-analyzer, the collision chamber and the ion source were well separated. In the region in which the electrons were detected the horizontal component of the fringing field of the magnet was the largest component; this component was perpendicular to the axis of the collision chamber. The other two components were less than five percent of this one. The measurement and guard electrodes were arranged in the collision chamber in such a way that the electric field between them was in approximately the same direction as the fringing field of the mass-analyzer. This configuration was chosen to reduce the possibility of displacing electrons, produced as a result of negative ion stripping, along the chamber axis.

3. METHOD OF MEASUREMENT

It was the purpose of the present measurements to obtain accurate data which would make possible a determination of the total effective cross section for negative-ion stripping in collisions with gas atoms. To realize this objective a strong electric field was set up between electrodes *1, 2, 3, 4* and *5, 6, 7, 8* to collect electrons which had been detached from negative ions along a definite path in the gas. Knowing the current *i* due to electrons produced in the gas along a path *l*, the current *I* in the negative ion beam, and the number of atoms *n* per cm^3 in the collision chamber, using the pressure and temperature of the gas it is possible to calculate the magnitude of the total cross section *Q* for the negative-ion stripping effect by the approximate formula

$$Q = i / Inl. \qquad (1)$$

The current *i* was measured by connecting the electrometer amplifier EMA-1 to one of the electrodes *2, 3,* or *6, 7*. In this case, the electrodes located in line with the measuring electrode were grounded and a negative potential was applied to all the opposite electrodes. A series of voltage-current curves which were taken indicated that a potential of −6 volts was sufficient in all cases to obtain the saturation current. The quantity *I* in Eq. (1) is the value of the current in the negative-ion beam on entrance into the measurement volume. To determine this quantity it is necessary to add the primary beam current at the side electrodes due to electrons produced by negative-ion stripping along the entire path from the entrance of the measurement chamber up to the collector.

The length *l* from which electrons are collected is taken to be the length of the measuring electrode. It was established in a series of special experiments that the ratio *i/I* was the same, within the limits of experimental accuracy, when any one of the four electrodes *2, 3, 6, 7* was used as a measuring electrode; this is an indication that edge effects were insignificant.

The gas pressure in the collision chamber was measured with an ionization gauge which was calibrated against a McLeod gauge.

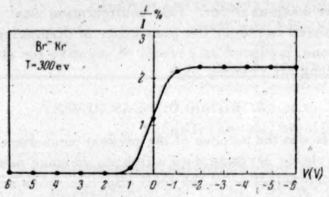

FIG. 2. Voltage-current curve. Br^- ions at 300 ev on krypton with $p = 2.6 \times 10^{-4}$ mm Hg.

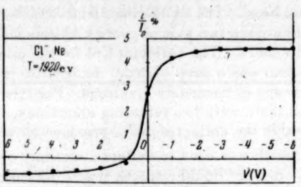

FIG. 3. Voltage-current curve. Cl^- ions at 1920 ev on neon with $p = 8.0 \times 10^{-4}$ mm Hg.

The various spurious effects which can distort the results obtained with the scheme described here have been considered in Ref. 1. We have carried out a series of control experiments to evaluate these effects under the conditions of the present experiment. By passing a beam of negative ions through the collision chamber when it was pumped down to a pressure of 1×10^{-6} mm Hg it was found that the beam was not intercepted by any of the side electrodes even when a potential difference of ±6 volts was applied between opposite pairs of electrodes. It was also verified that secondary electrons from the edges of aperture A_2 and from the inner surface of the collector Col did not enter the measurement volume. When the collision chamber was filled with gas it was found that electrons produced as a result of negative-ion stripping within the collector did not enter the measurement volume whether or not a potential was applied to electrodes *9* and *10* inside the collector.

In all (negative ion and gas atom) pairs which were investigated, voltage-current curves $i(V)$ (where V is the potential of the electrode opposite the measuring electrode) were measured for the highest, lowest and one intermediate value of the ion energy, while V was varied from +6 volts to −6 volts. At low ion energies (<500 to 600 ev) the voltage-current curves were similar to that shown in Fig. 2 which pertains to a Br^- ion at an energy $T = 300$ ev and atomic Kr. In the region of the field which was used to accelerate electrons toward the measurement electrode, saturation current was achieved at $V \approx -2$ to -3 volts. In the retarding field region the current to the measuring electrode practically vanished at $V = +2$ volts. The absence of a negative current in the left side of Fig. 2 is a consequence of the fact that the negative ions are scattered in the gas and do not reach the measurement electrode in any appreciable quantity; this situation may be explained by the small probability for charge-conserving scattering of negative ions at large angles.

In a number of cases in which the ion energy was greater than 500 to 600 ev a small positive current was observed in the left side of the voltage-current curve. A typical case, in which this effect was particularly large, is the voltage-current curve shown in Fig. 3 which was obtained in collisions of Cl^- ions at energy $T = 1920$ ev with Ne atoms. The appearance of a positive current might be explained by the production of positive ions in the passage of negative ions through the gas or by the ejection of electrons from the surface of the measurement electrode, for example, as a result of the emission of radiation in the collision of the negative ions with the gas atoms. In this and similar cases, in determining the effective cross section for negative-ion stripping the current i was taken as the difference of the negative and the positive saturation currents. This was done in the present case when the positive current was not small compared with the negative current*.

The possibility that there might be errors in the present results because of the superposition, on the negative-ion stripping, of an exchange effect with atoms of the gas is obviously negligible because the inert gases, in which the bulk of the measurements were made, have zero electron affinity. The absence of charge exchange effects can be con-

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* To investigate the origin of this positive current, an experiment was performed in which an open grid was placed in front of the measuring electrode and slow positive ions were produced in the collision chamber (by charge exchange between Cl^+ ions in Kr). The dependence of the positive current to the measurement electrode on the grid potential was noticeably different in this case from the dependence of the positive current observed in the ion-stripping case, Cl^- in Br. On the basis of these results it may be assumed that the origin of the positive current in the latter case is the ejection of electrons from the measurement electrode.

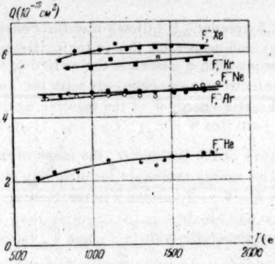

FIG. 4. Effective cross section for stripping of the negative ion F^- in inert gases as a function of the ion energy.

clusively demonstrated only in the hydrogen experiments. However, if the H_2 molecule has an electron affinity, the probability for charge exchange between negative halogen ions and molecular H_2 must still be small because of the great difference in the magnitude of the electron bonding energy.

In all cases in which the voltage-current characteristics were taken the dependence of the ratio i/I on the pressure of the gas p in the collision chamber was also investigated (up to a pressure of $\approx 1 \times 10^{-3}$ mm Hg). The measurements which were used to determine the cross section Q were performed at the values of p which pertain to the linear part of the curve $(i/I)(p)$ in which the magnitude of i/I is less than five percent. Under these experimental conditions the current attributed to negative-ion stripping in the absence of a gas in the apparatus amounted to 0.1 to 0.2 percent of the current produced by the ion stripping in the gas being studied.

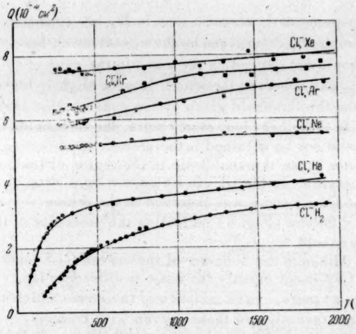

FIG. 5. Effective cross section for stripping of the negative ion Cl^- in inert gases and hydrogen as a function of ion energy.

The reproducibility of the measured results was quite good: The average random error in the determination of Q was no greater than six percent. Systematic errors in the measurement of the gas pressure and the length from which electrons were collected in the collision chamber and the use of the approximate formula (1) may increase the total error in the determination of the absolute value of Q to 15 to 20 percent.

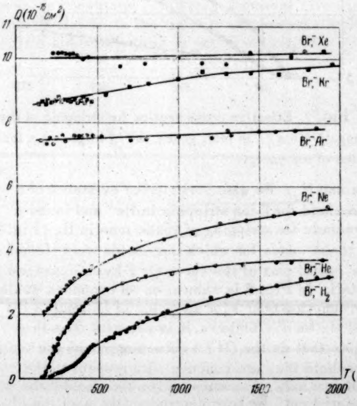

FIG. 6. Effective cross section for stripping of the negative ion Br^- in inert gases and hydrogen as a function of ion energy.

4. RESULTS

The measurements which were carried out in the present work have been used to determine the total effective cross sections Q for negative-ion stripping for all the halogens in all the inert gases in the ion-energy region from approximately 200 to 2,000 ev. The quancity Q was also determined for collisions of Cl^-, Br^- and I^- uons with H_2 molecules. The results which were obtained are presented in Figs. 4-7 as curves giving the dependence of Q on the ion kinetic energy T.

We were also able to find the threshold for stripping (Fig. 6) in the case of the Br^- ion in

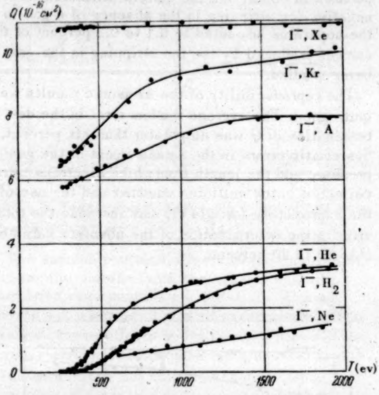

FIG. 7. Effective cross section for stripping of the negative ion I⁻ in inert gases and hydrogen as a function of ion energy.

He and H_2. We also verified the existence of a threshold for I⁻ ion stripping in He[1] and found a threshold for stripping of these ions in H_2 (Fig. 7). In those cases for which thresholds were found, the initial part of the curve $Q(T)$ was examined in detail. In Fig. 8 is shown, on an expanded scale, the function $Q(T)$ for stripping of I⁻ ions in He and H_2 for $T < 600$ ev. It is apparent from this figure that as the $Q(T)$ curves approach the threshold they are concave with respect to the ordinate axis, thus causing the threshold to be smeared out. We have determined the position of the threshold by square-law extrapolation of the initial part of the experimental curves. The uncertainty in the value of T because of the finite resolving power of the mass-analyzer is no greater than ±2.5 percent. It follows from the conservation of energy and momentum that in the collision of a negative ion with a gas atom the stripping of the excess electrons can occur only with the consumption of kinetic energy W of the relative motion of the two particles $W = m_2 T/(m_1 + m_2)$, where m_1 is the mass of the ion and m_2 the mass of the gas atom. The energy threshold for stripping should occur at $W = W_0 = S$, where S is the binding energy of the excess electron in the negative ion. In the Table are shown the values T_0 and W_0 for threshold obtained from the present experimental curves and, for comparison, S, the electron bonding energy in the atoms Br and I. As can be seen from the Table, the observed thresholds lie above the thresholds determined from the conservation of energy and momentum. A similar result was obtained earlier in Ref. 4; there, however, an explanation for the origin of the experimentally observed threshold was proposed. It should be noted that the thresholds W_0 for ion stripping for Br⁻ and I⁻ in He lie higher than the thresholds for the stripping of these same ions in H_2, whereas the thresholds determined by the conservation law should not depend on the gas. In the case of Br⁻, H_2, the observed threshold is only slightly higher than the threshold given by the conservation laws.

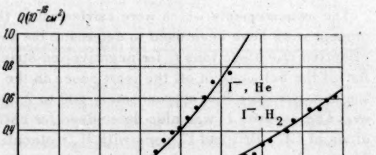

FIG. 8. Initial part of the curves $Q(T)$ for stripping of the negative ion I⁻ in He and H_2.

TABLE

Ion Atom	T_0, ev	W_0, ev	S, ev
Br⁻, He . . .	150	7.2	3.6 [6]
Br⁻, H_2	160	3.9	
I⁻, He	280	8.6	3.3 [7]
I⁻, H_2	320	5.0	

In the other (ion, atom) pairs, the thresholds could not be obtained in the present experiments, since these thresholds lie in the region of low ion energies. In the Cl⁻ in He case a rapid drop of the $Q(T)$ curve was observed in the region $T < 200$ ev (Fig. 5) indicating the proximity of the threshold.

Although the behavior of the curves $Q(T)$ and $Q(W)$ is not exactly the same in different (ion, atom) pairs, in the majority of the cases which were investigated these curves have common features. In Fig. 9 is presented the curve $Q(W)$ for the stripping of I⁻ in He which will be taken as typical. Close to the threshold the cross section increases rather rapidly with W, then the slope

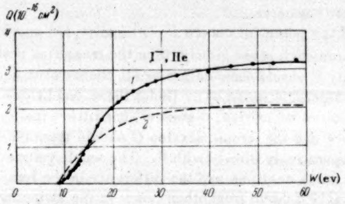

FIG. 9. The functions $Q(W)$ and $P(W)$ for stripping of the negative ion I⁻ in He. *1* – $Q(W)$ determined experimentally; *2* – $P(W)$ as calculated from Eq. (3).

falls off and the value of the cross section approaches a certain limiting value Q_m. In those cases in which the energy threshold for negative-ion stripping should be found at small ion energies (for example, collisions with Kr and Xe atoms) we were able to observe only the segment of the curve in which the value of Q was sensibly constant.

The present data reveal a correlation between the quantity Q_m and the atomic numbers of the ion and atom between which the collision takes place. For a given ion, for example Cl⁻ (Fig. 5), Q_m increases in going from He to Ne and then further from A, Kr and Xe. In collisions with the same atom, for example Xe, Q_m increases with increasing atomic number of the ion. A marked departure from this dependence was observed in the case of I⁻, Ne for which the $Q(T)$ curve passes below the curve for I⁻ in the energy region 600 to 1,900 ev and does not saturate. This peculiarity was investigated further in supplementary measurements of the cross section for ion stripping of Sb⁻ and Te⁻ (the mass of which is close to the mass of the I⁻ ion) in Ne and He. The same shape for the corresponding curves was found for these ions as for the I⁻ ion. In Fig. 10 are shown curves for $Q(W)$ in He and Ne for I⁻ and Te⁻ ions.

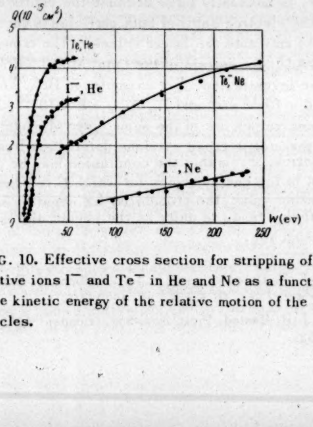

FIG. 10. Effective cross section for stripping of the negative ions I⁻ and Te⁻ in He and Ne as a function of the kinetic energy of the relative motion of the particles.

In a number of cases the present results can be compared with the data of Refs. 1-3. All the negative-ion stripping cross sections measured in the present work are smaller than the corresponding values given in Ref. 1. This difference may be explained by the more favorable experimental conditions in the present work as compared with Ref. 1. The general shape of the $Q(T)$ curves in the present work are similar to the curves presented in Refs. 2 and 3. The absolute value of Q, however, is considerably different from that given in Refs. 2 and 3. In the majority of cases the present values are smaller, and in only one pair, Cl⁻, Ne, are the values in good agreement with those given in Ref. 3.

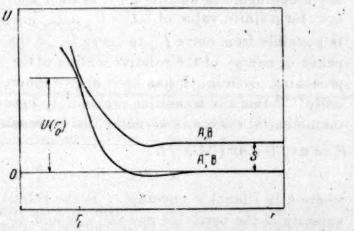

FIG. 11. Potential curves associated with the stripping of an electron from the negative ion A^- in slow collisions with the atom B.

5. DISCUSSION OF THE RESULTS

An approximate theoretical calculation (using the Born approximation) of the magnitude of the effective cross section for negative-ion stripping has been carried out in only one case, namely, the collision of H⁻ ions with He atoms[8]. For heavier atoms and ions a quantum-mechanical calculation of the cross sections would be prohibitively complicated. For cases in which the collision of an ion with a gas atom can be considered slow (the relative velocity of the approaching particles is smaller than the velocity of the electron in the negative ion), a qualitative description of the stripping process can be given in terms of a picture borrowed from the theory of diatomic molecules[1]. Two particles which approach and then separate can be considered as a short-lived system with a potential energy U which depends on the distance r between the nuclei. Assume that the curve $U_1(r)$ is associated with one system (a negative ion A^- and atom B) and the curve $U_2(r)$ is associated with the other system (atom A and atom B). At large r the curve $U_2(r)$ should lie above the curve $U_1(r)$ by an amount equal to the magnitude of S the electron bonding energy for atom A (Fig. 11). At smaller r, U_1 and U_2 both increase (because of the repulsive forces between the particles) and if there is no interaction between the systems (A^- and B) and (A, B and free electron), $U_1(r)$ and $U_2(r)$ can intersect at a certain $r = r_0$. If there is an interaction no intersection occurs; at the point r_0 the potential curves approach to within a minimum distance but become separated again at smaller r. It is well known that even for a finite value of $\Delta U = U_1 - U_2$ a transition is possible from curve U_1 to curve U_2 at the expense of energy of the relative motion of the approaching particle. It has been shown theoretically[9-11] that the transition probability between the potential curves is given by the expression

$$P = \exp\{-4\pi(\Delta U)^2/hv \times (dU_1^0/dr - dU_2^0/dr)\}, \tag{2}$$

where h is Planck's constant, v is the relative velocity of the particles and dU_1^0/dr and dU_2^0/dr are the values of the derivative dU/dr for each system in the absence of any interaction.

If the curves U_1 and U_2 are such that U is a strong function of r, then P will have a pronounced maximum at $r = r_0$ and the transition probability will be significant only at the critical distance r_0. It is easy to show that in this case the transition probability P and the effective cross section Q for the process associated with this transition are related by the expression

$$Q = P(r_0, W)\pi r_0^2(1 - U(r_0)/W), \tag{3}$$

where $U(r_0)$ is the energy of the relative particle motion corresponding to the threshold for the process in question. The validity of Eq. (3) in determining the behavior of the probability for negative-ion stripping P as a function of ion energy can be determined from the experimental data on the function $Q(W)$. This was done for stripping of I^- ions in collision with He atoms [$U(r_0) = W_0$ is unknown in this case] using the experimental curve given in Fig. 9 and the function $P(W)$ given in this same figure as curve 2.

If the potential curves $U_1(r)$ and $U_2(r)$ approach each other slowly, then the transition probability P should show only a small change over a considerable range of r. Under these conditions the notion of a "critical distance" has little significance and the cross section Q should increase comparatively slowly with W. The small values of the cross sections and the difference in the form of the $Q(W)$ curve from other cases in the stripping of I^-, Te^- and Sb^- ions in neon may possibly be explained on this basis.

It follows from Eq. (3) that for $W \gg U(r_0)$, $Q = P(r_0 W)\pi r_0^2$. As has already been noted in Sec. 4, in many (ion, atom) pairs virtually constant values of the cross section Q were obtained in a certain region of ion energy. This situation is an indication that the probability for electron detachment $P(r_0, W)$ is a weak function of W in these regions.

The correlation which was observed between the quantities Q_m and the atomic numbers of the ion and atom (Sec. 4) may be reasonably explained by the hypothesis that the value of r_0 is different for different pairs and is connected with the "size" of the colliding particles. It is well known that the "radii' of atoms and negative ions increase with increasing atomic number. This correlation can, however, be disturbed by other factors which influence the value of r_0; such factors might be the polarizability of the interacting particles or some peculiarity of the structure of the electron shell. Thus, for example, for F^- ions the Q_m in neon and A are almost identical (cf. Fig. 4). It is impossible that for the pair F^-, Ne the value of r_0 is unusually large because the configuration of the electron shell of both particles is the same.

We may note the larger values of the cross sections Q_m for certain of the (ion, atom) pairs which were investigated. For example, in Br^-, Xe, $Q = 1.1 \times 10^{-15}$ cm^2 and I^-, A, $Q = 8 \times 10^{-16}$ cm^2. These values are of the same order of magnitude as the atomic cross sections determined from gas kinetics. From this the conclusion may be drawn that in the energy region for which Q reaches a limiting value, the probability for negative-ion stripping tends to unity as the impact distances approach r_0.

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On the Theory of Atomic Semiconductors

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On the basis of Vonsovskii's polar model of crystals several questions in the theory of absorption of light and the theory of photoconductivity in atomic semiconductors are examined, taking excitons into account. We consider terms of third order in the Hamiltonian, which determine the probability of different transitions among the elementary forms of excitation; this allows an examination of the kinetics of photoconductivity.

1. INTRODUCTION

THE polar model of crystals was proposed in 1934 by Shubin and Vonsovskii[1]. In this model the semiconductor in the normal state is considered as an ideal monocrystal, at the lattice sites of which are atoms with one valence s-electron. In the excited state sites can occur at which there are two electrons, and correspondingly, empty sites. The appearance of this type of excited state causes its electrical conductivity. Actually, on account of the translational symmetry, the states of the sites in which there are two electrons (or, respectively, none) can propagate through the crystal, which leads to the appearance of degenerate states. Since each state with definite sites occupied by two electrons, or with empty sites, is quasi-stationary, we obtain a whole band of energy levels. In this way the energy spectrum of an atomic semiconductor, from the point of view of the polar model, can even be continuous; but since the excitation of states with double and empty sites requires an expenditure of energy, the excitation of current states in the crystal requires a known activation energy, in spite of the existence of the continous spectrum. Thus, for example, if one considers the exchange interaction between the electrons, the energy spectrum of the crystal has the form shown in Fig. 1, from which it can be seen that the conduction states occurring in the upper band, overlapping the lower one (due to the exchange interaction), require an activation energy for their excitation.

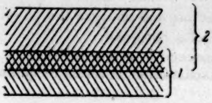

FIG. 1. *1* – Nonconduction band; *2* – Conduction band.

It is known[2] that an energy spectrum of this form (for weak excitation of the crystal) can be regarded as the energy spectrum of a system of noninteracting quasi-particles. In particular, the branch of the energy spectrum corresponding to the existence of conduction states in the crystal can be considered as the energy spectrum of a collection of positively charged (holes) and negatively charged (doublets) quasi-particles.

Besides the types of excitation of the crystal mentioned above there are states possible in which the electron in one or several atoms is found in an excited state. To such states, on account of translational degeneracy, there also corresponds a definite energy band. The corresponding quasi-particles are called Frenkel excitons. It is not difficult to show by a simple calculation (this, by the way, is also obvious physically) that the Frenkel excitons form currentless states (i.e., these quasi-particles are electrically neutral).

In what follows we shall neglect the exchange energy. This leads to the result that the lower energy band of the spectrum represented in Fig. 1 and corresponding to the currentless states degenerates into a single energy level, separated by a gap from the band of excited states. Assuming that the excitation energy of the excitons is greater than the excitation energy of the conduction states, we obtain an overlapping (in the upper part) of the exciton energy band and the energy band corresponding to the conduction states (Fig. 2).

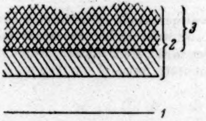

FIG. 2. *1* – Nonconducting state; *2* – Conduction band; *3* – Exciton band.

2. THE HAMILTONIAN OF AN ATOMIC SEMICONDUCTOR ON THE POLAR MODEL

We shall introduce the basic features of the derivation of the Hamiltonian for the polar model of semiconductors and analyze the physical meaning of the results obtained.

We shall proceed from the well-known representation of the Hamiltonian in terms of quantized wave functions:

$$H = \int \Psi^{+}(x)\, H(x) \Psi(x)\, dx \tag{2.1}$$

$$+ {}^1/_2 \int \Psi^{+}(x) \Psi^{+}(x')\, G(|\mathbf{r} - \mathbf{r}'|) \Psi(x')\, \Psi(x)\, dx dx',$$

$$H(x) = -\frac{h^2}{2m} \nabla^2 + \sum_q G(|\mathbf{r} - \mathbf{R}_q|) \tag{2.2}$$

is the Hamiltonian of an electron moving in the force field of all the ions; $G(|\mathbf{r} - \mathbf{R}_q|)$ is the potential of an ion located at site q;

$$G(|\mathbf{r} - \mathbf{r}'|) = e^2 / |\mathbf{r} - \mathbf{r}'| \tag{2.3}$$

is the potential energy of interaction of two electrons. The quantized wave functions have the form

$$\Psi(x) = \sum_k a_k \psi_k(x); \quad \Psi^{+}(x) = \sum_k a_k^{+} \psi_k^{*}(x), \tag{2.4}$$

where a_k^{+}, a_k are the usual operators of the second quantization for the electrons (Fermi amplitudes), $\psi_k(x)$ is a complete orthogonalized system of functions; the summation is taken over all possible states of the electron. Substituting (2.4) into (2.1) we obtain the well-known expression for the Hamiltonian of a system of electrons in terms of Fermi operators

$$H = \sum_{kl} (k \mid H \mid l)\, a_k^{+} a_l \tag{2.5}$$

$$+ \frac{1}{2} \sum_{klps} (kl \mid G \mid ps)\, a_k^{+} a_l^{+} a_s a_p.$$

The matrix elements are determined by the expressions

$$(k \mid H \mid l) = \int \psi_k^{*}(x)\, H(x)\, \psi_l(x)\, dx, \tag{2.6}$$

$$(kl \mid G \mid ps) \tag{2.7}$$

$$= \int \psi_k^{*}(x)\, \psi_l^{*}(x')\, G(|\mathbf{r} - \mathbf{r}'|)\, \psi_p(x)\, \psi_s(x')\, dx\, dx'.$$

In the polar model of atomic semiconductors one proceeds from the following approximation. The operator (2.5) is taken as being the Hamiltonian, but in calculating the matrix elements, the atomic wave functions, i.e., the wave functions of the isolated atoms, are used. Here, if it is necessary to consider the existence of exciton states, not only the wave functions of the normal state but also the wave functions of the first excited state are taken into account. This system of functions is not complete.

Furthermore, functions referring to different atoms are nonorthogonal. In this way, strictly speaking, the expression for the Hamiltonian (2.5), computed by use of atomic functions, is inexact. However, considering that the wave functions of different atoms overlap weakly, their nonorthogonality is neglected in the polar theory of semiconductors. It would be more nearly correct to orthogonalize the system of atomic wave functions by the method of Bogoliubov[3]; however, this will not be done in this work. We should have a complete system of functions if we considered all possible excited states of the atoms. The incompleteness of the system of functions used apparently does not have any essential significance in the majority of cases.

Thus we put

$$\psi_k(x) = \varphi_i(\mathbf{r} - \mathbf{R}_q)\, u_t(\sigma), \tag{2.8}$$

where $i = 0$ corresponds to the ground state, $i = 1$ to the excited state of the atom, $u_t(\sigma)$ is the spin function [in Eqs. (2.1)-(2.7) we understand by x the collection of space and spin-coordinates]. Calculating the matrix elements by means of such functions and substituting them into (2.5) we obtain, after summing over spins,

$$H = \sum H_{q_1 i_1;\, q_2 i_2} a^+_{q_1 i_1 s} a_{q_2 i_2 s} \tag{2.9}$$

$$+ \frac{1}{2} \sum I_{q_1 i_1;\, q_2 i_2;\, q_3 i_3;\, q_4 i_4} a^+_{q_1 i_1 s} a^+_{q_2 i_2 s'} a_{q_4 i_4 s'} a_{q_3 i_3 s'}$$

$$H_{q_1 i_1;\, q_2 i_2} = \int \varphi^*_{i_1}(\mathbf{r} - \mathbf{R}_1)\, H(r)\, \varphi_{i_2}(\mathbf{r} - \mathbf{R}_{q_2})\, d\tau \tag{2.10}$$

[$H(r)$ in fact coincides with $H(x)$, since the spin variables do not enter into the last expression].

$$I_{q_1 i_1;\, q_2 i_2;\, q_3 i_3;\, q_4 i_4} \tag{2.11}$$

$$= \int \varphi^*_{i_1}(\mathbf{r} - \mathbf{R}_{q_1})\, \varphi^*_{i_2}(\mathbf{r}' - \mathbf{R}_{q_2})$$

$$\times G(|\mathbf{r} - \mathbf{r}'|)\, \varphi_{i_3}(\mathbf{r} - \mathbf{R}_{q_3})\, \varphi_{i_4}(\mathbf{r}' - \mathbf{R}_{q_4})\, d\tau\, d\tau'.$$

As is well known, the Fermi operators a^+_{qis} and a_{qis} act on wave functions, the arguments of which are the occupation numbers of the electronic states $C(n_{qis})$. In the polar model the states of the crystal are characterized by the states of the lattice sites. Let us introduce the symbols

$$R_q = \begin{cases} 1 \text{ when one electron with a right-hand spin is at site } q, \\ 0 \text{ in all remaining cases} \end{cases}$$

and analogously, L_q for an electron with a left-hand spin, E^+_q for an empty site, E^-_q for a site with two electrons, E^L_q and E^R_q for an excited electron with left- and right-hand spins, respectively.

Clearly, the equation

$$R_q + L_q + E^+_q + E^-_q + E^R_q + E^L_q = 1 \tag{2.12}$$

must be satisfied. The quantities R_q, L_q, E^+_q can be taken as the dynamical variables of our system, having the meaning of the corresponding occupation numbers. The wave functions then must be considered as functions of these variables.

Let us introduce the operators of the second quantization corresponding to these variables: φ_q, χ_q, Ψ_q, Φ_q, ζ_q, ϑ_q (and those associated with them) act respectively on the variables R_q, L_q, E^+_q, E^-_q, E^R_q, E^L_q. Following Shubin and Vonsovskii[1] we assume that these operators obey the Bose commutation relations.

For the transformation of the Hamiltonian (2.9) to these operators, we proceed in the following way. We designate by $C_0 = C(1, 0; 1, 0; 1, 0; \ldots)$ the wave function of the "zero-order" state (one electron with a left-hand spin on each atom). Then a function of the form

$$C = \prod_{i=1}^{s} C^+_{g_l} a^+_{f_i} \prod_{l=1}^{u} C^+_{h_l} a^+_{h_l} \varepsilon(\ldots; 1,0; \ldots)\, C_0 \tag{2.13}$$

$$= \varepsilon(f_1 \ldots f_s;\, g_1 \ldots g_s;\, r_1 \ldots r_m;\, h_1 \ldots h_n)$$

$$\times \prod_{i=1}^{s} C_{g_i} a^+_{f_i} \prod_{l=1}^{u} C{+}l_{h_l} a^+_{h_l} C_0$$

corresponds to the state of the crystal in which at sites $f_1, \ldots f_s$ there are two electrons each (with opposite spins), the sites $g_1, \ldots g_s$ are empty, at the sites $k_1, \ldots h_n$ there is one electron with a right-hand spin, and at the remaining sites $r_1, \ldots r_m$ there is one electron with a left-hand spin. Here $C_{gi} = a^+_{gi, -½}$; $a^+_{fi} = a^+_{fi, ½}$. The function

$$\varepsilon(f_1 \ldots f_s;\, g_1 \ldots g_s;\, r_1 \ldots r_m;\, h_1 \ldots h_n) \tag{2.14}$$

$$= \varepsilon(\ldots;\, n^L_q,\, n^R_q;\, \ldots)$$

is symmetric in the indices r or h separately and antisymmetric in the indices f or g.

Noting that $C(\ldots n_q, m_q; \ldots) = D(\ldots, R_q, L_q, E_q^+, E_q^-, E_q^R, E_q^L, \ldots)$, it is possible to relate uniquely the result of the action of the Bose operators on the function D to the result of the action of the Fermi operator on the function C. Thus, for example, it is easy to convince oneself that to the operator $a_q^+ C_q C_q a_{q'}$ corresponds the operator $\psi_q \Phi_q^+ \Phi_{q'} \psi_{q'}^+$.

Omitting all these transformations, we write down immediately the expression for the Hamiltonian in Bose operators without taking into account the excited states of the atom, which is considerably simpler than the complete Hamiltonian, and we show by this example which simplifications are to be used in the sequel. It must also be mentioned that the expression presented below for the Hamiltonian already contains a series of simplifications, inasumch as terms were discarded which correspond to the simultaneous transition of two or more electrons (the integrals corresponding to such transitions are considerably smaller than the remaining terms). Therefore, for the Hamiltonian in the polar model (without taking into account exciton states) we obtain the well-known Hamiltonian of Vonsovskii[1]

$$
\begin{aligned}
H = {}& {}^1/_2 (A+D) \sum (\Psi_q^+ \Psi_q + \Phi_q^+ \Phi_q) \\
& + {}^1/_2 \sum B_{qq'} (\Psi_q^+ \Psi_q - \Phi_q^+ \Phi_q)(\Psi_{q'}^+ \Psi_{q'} - \Phi_{q'}^+ \Phi_{q'}) \\
& + {}^1/_2 \sum I_{qq'} (\Phi_q^+ \Psi_{q'}^+ - \Psi_q^+ \Phi_{q'}^+)(\Phi_q \Psi_{q'} - \Psi_q \Phi_{q'}) \\
& - {}^1/_2 \sum I_{qq'} (\Psi_q^+ \Psi_q + \Phi_q^+ \Phi_q)(\Psi_{q'}^+ \Psi_{q'} + \Phi_{q'}^+ \Phi_{q'}) \\
& + {}^1/_2 \sum I_{qq'} (\chi_{q'}^+ \varphi_q^+ - \chi_q^+ \varphi_{q'}^+)(\chi_{q'}^+ \varphi_q - \chi_q \varphi_{q'}) \\
& + {}^1/_2 \sum L_{qq'} [(\varphi_q \varphi_{q'}^+ + \chi_q \chi_{q'}^+)(\Phi_q^+ \Phi_{q'} - \psi_q^+ \Psi_{q'}) \\
& + (\varphi_{q'} \varphi_q^+ + \chi_{q'} \chi_q^+)(\Phi_{q'}^+ \Phi_q - \Psi_{q'}^+ \Psi_q)] \\
& + {}^1/_2 \sum L_{qq'} [(\varphi_q \chi_{q'} - \chi_q \varphi_{q'})(\Phi_q^+ \Psi_{q'}^+ \\
& - \Phi_{q'}^+ \Psi_q^+) + (\varphi_q^+ \chi_{q'}^+ - \chi_q^+ \varphi_{q'}^+)(\Phi_q \Psi_{q'} - \Psi_q \Phi_{q'})]
\end{aligned}
\tag{2.15}
$$

(the summations are over all indices q, q').

In what follows we shall examine cases in which the number of excitations is small; in other words, in which at the majority of lattice sites simple atoms with right- and left-hand spins occur, whereby both directions of spin are considered equally probable. Therefore, $\overline{\varphi_q^+ \varphi_q} \sim 1/2$ and $\overline{\chi_q^+ \chi_q} \sim 1/2$, where the bar designates the average value. Since the number of simple atoms is considerably larger than the number of empty and double sites, we can neglect the changes of state of the collection of simple atoms in the occurrence of these and other excitations; and we can consider this collection of simple atoms as a classical system, a special reservoir, from which can arise empty and double sites, but the states of which do not change thereby. This is somewhat analogous, on the one hand, to the introduction of a thermostat in the derivation of the Gibbs distribution; or, on the other hand, it is analogous to the classical representation of radiation of great intensity in the quantum representation of the absorbing atom. In the latter case the radiation is considered classically and is described by commuting quantities. Analogously, in our case, considering the collection of simple atoms as a classical system, we can consider operators referring to the simple atoms as commuting quantities, and in correspondence with this we put

$$
\varphi_q^+ = \varphi_q = \chi_q^+ = \chi_q = 2^{-1/2}. \tag{2.16}
$$

Hence, the Hamiltonian (2.15) simplifies and takes the following form:

$$
\begin{aligned}
H = {}& {}^1/_2 (A+D) \sum (\Psi_q^+ \Psi_q + \Phi_q^+ \Phi_q) \\
& + {}^1/_2 \sum B_{qq'} (\Psi_q^+ \Psi_q - \Phi_q^+ \Phi_q)(\Psi_{q'}^+ \Psi_{q'} - \Phi_{q'}^+ \Phi_{q'}) \\
& + {}^1/_2 \sum I_{qq'} (\Phi_q^+ \Psi_{q'}^+ - \Psi_q^+ \Phi_{q'}^+)(\Phi_q \Psi_{q'} - \Psi_q \Phi_{q'}) \\
& - {}^1/_2 \sum I_{qq'} (\Psi_q^+ \Psi_q + \Phi_q^+ \Phi_q)(\Psi_{q'}^+ \Psi_{q'} + \Phi_{q'}^+ \Phi_{q'}) \\
& + {}^1/_2 \sum L_{qq'} [\Phi_q^+ \Phi_{q'} + \Phi_{q'}^+ \Phi_q - \Psi_q^+ \Psi_{q'} - \Psi_{q'}^+ \Psi_q].
\end{aligned}
\tag{2.17}
$$

In this approximation the principal feature is the circumstance that the operators of creation and annihilation of empty and double sites disappear, so that in this approximation the average number of the latter is an integral of the motion. Further, an additional approximation occurs, which results in the fact that in the operator (2.17) only terms quadratic in the operators remain, i.e., the energy of mutual interaction of the excitations is neglected, considering the number of excitations small. Then in place of (2.17), we obtain

$$H = {}^1/_2 (A + D) \sum (\psi_q^+ \psi_q + \Phi_q^+ \Phi_q) \qquad (2.18)$$
$$+ {}^1/_2 \sum L_{qq'} (\Phi_q^+ \Phi_{q'} + \Phi_q^+ \Phi_q - \Psi_q^+ \Psi_{q'} - \Psi_{q'}^+ \Psi_q).$$

The expression for the Hamiltonian, considering the exciton states of the atoms and corresponding to (2.15), is too cumbersome, and we shall not present it here. Below is given the expression for the Hamiltonian, taking into account the exciton states, in the approximation corresponding to (2.18), with, however, this difference: that in it are included terms of the third order relative to the operators of second quantization:

$$H = H_{(2)} + H_{(3)}, \qquad (2.19)$$

$$(2.20)$$
$$H_{(2)} = Q^{(1)} \sum (\alpha_q^+ \alpha_q + \beta_q^+ \beta_q)$$
$$+ Q^{(2)} \sum (\Psi_q^+ \Psi_q + \Phi_q^+ \Phi_q)$$
$$+ \sum Q_{fq}^{(3)} (\alpha_q^+ \alpha_f + \alpha_q \alpha_f^+) + \sum Q_{fq}^{(3)} (\beta_q^+ \beta_f + \beta_q \beta_f^+)$$
$$+ \sum Q_{fq}^{(4)} (\Phi_q^+ \Phi_f + \Phi_f^+ \Phi_q)$$
$$- \sum Q_{fq}^{(4)} (\Psi_q^+ \Psi_f + \Psi_q \Psi_f^+);$$

$$(2.21)$$
$$H_{(3)} = \sum N_{fq}^{(1)} [\psi_q^+ \psi_f (\alpha_q - \beta_q) + \psi_q \psi_f^+ (\alpha_q^+ - \beta_q^+)]$$
$$+ \sum N_{fq}^{(2)} [\Phi_q^+ \Phi_f (\alpha_q - \beta_q) + \Phi_q \Phi_f^+ (\alpha_q - \beta_q)]$$
$$+ \sum N_{fq}^{(3)} [\Psi_q^+ \Phi_f^+ (\alpha_q + \beta_q) + \Psi_q \Phi_f (\alpha_q^+ + \beta_q^+)]$$
$$+ \sum N_{fq}^{(4)} [\Phi_q^+ \Psi_f^+ (\alpha_q + \beta_q) + \Phi_q \Psi_f (\alpha_q^+ + \beta_q^+)]$$
$$+ \sum N_{fq}^{(5)} (\beta_q^+ \beta_q + \alpha_q^+ \alpha_q) (\alpha_f + \alpha_f^+ - \beta_f - \beta_f^+)$$
$$+ \sum N_{fq}^{(6)} (\Psi_f^+ \Psi_f - \Phi_f^+ \Phi_f) (\alpha_q + \alpha_q^+ - \beta_q - \beta_q^+)],$$

Q and N are certain coefficients, the form of which we shall not write down in the present article.

The operators α_q, α_q^+, β_q^+, β_q^+ are the operators for the left-hand and right-hand excited sites (exciton sites) which are connected with the operators of the excited sites presented earlier by the relations

$$\vartheta_q = \alpha_q + c, \quad \vartheta_q^+ = \alpha_q^+ + c, \qquad (2.22)$$
$$\zeta_q = \beta_q - c, \quad \zeta_q^+ = \beta_q^+ - c,$$

where c is a certain constant, chosen in a definite way. It is necessary to carry out the transformation (2.22) in order that terms in the Hamiltonian of first order relative to the operators disappear.

The physical meaning of the quadratic and third order terms in the operators is more easily explained after the transformation of the Hamiltonian to the space of quasi-momenta and the introduction of the corresponding quasi-particles.

3. TRANSFORMATION TO THE SPACE OF QUASI-MOMENTA

For the transition to the space of quasi-momenta we introduce the following canonical transformation:

$$\Psi_q = N^{-1/2} \sum_k \Psi_k e^{-i(\mathbf{k} - \pi \mathbf{b}) \mathbf{R}_q}; \qquad (3.1)$$
$$\Phi_q = N^{-1/2} \sum_k \Phi_k e^{-i\mathbf{k}\mathbf{R}_q},$$
$$\alpha_q = N^{-1/2} \sum_k \vartheta_k e^{-i\mathbf{k}\mathbf{R}_q},$$
$$\beta_q = N^{-1/2} \sum_k \zeta_k e^{-i\mathbf{k}\mathbf{R}_q},$$

where **b** is a vector of the reciprocal lattice (and analogously for those associated with it). Operators with the indices k, describing processes of creation or annihilation of the corresponding quasi-particles, satisfy the same commutation rules as the operators with indices q. In the Fourier transformation (3.1), the writing of $(\mathbf{k} - \pi \mathbf{b})$ in place of $\mathbf{k}$ for the operator Ψ denotes that the subtraction of the quasi-momenta of the holes is carried out from the upper edge of the band, and not from the lower one, as is done for the doublets and excitons. Thanks to this, we obtain a positive effective mass for the holes.

After carrying out the transformation, we obtain

$$(3.2)$$
$$H_{(2)} = \sum_k E'_k n_k^{\vartheta} + \sum_k E'_k n_k^{\zeta} + \sum_k E_k n_k + \sum_k E_k n_k^{\Psi},$$

where $n_k^{\vartheta} = \vartheta_k^+ \vartheta_k$, n_k^{ξ}, n_k^{Ψ} are operators for the occupation numbers of the left-hand excitons, right-hand excitons, doublets and holés. In this way the Hamiltonian (3.2) is presented in the form of the sum of the energies of the elementary excitations. E'_k, E_k have the meaning of the energies of the corresponding elementary excitations. In the effective-mass approximation they have the form

$$E_k = \Delta E + \hbar^2 k^2 / 2\mu, \; E'_k = \Delta E' + \hbar^2 k^2 / 2\mu', \tag{3.3}$$

where ΔE is the excitation energy of a doublet or hole, $\Delta E'$ is the energy of excitation of an exciton, μ is the effective mass of a doublet (hole), μ' is the effective mass of an exciton.

As far as the terms of third order are concerned, after transformation to the space of quasi-momenta we obtain

$$\begin{aligned} H_{(3)} = {} & \Sigma K^{(1)}(\mathbf{k},\ \mathbf{k}'')\,\delta(\mathbf{k}' - \mathbf{k} - \mathbf{k}'') \\ & \times [\Psi^+_{k'}\Psi_{k''}(\vartheta_k - \zeta_k) + \Psi_{k'}\Psi^+_{k''}(\vartheta^+_k - \zeta^+_k)] \\ & + \Sigma K^{(2)}(\mathbf{k},\ \mathbf{k}'')\,\delta(\mathbf{k}' - \mathbf{k} - \mathbf{k}'') \\ & \times [\Phi^+_{k'}\Phi_{k''}(\vartheta_k - \zeta_k) + \Phi_{k'}\Phi^+_{k''}(\vartheta^+_k - \zeta^+_k)] \\ & + \Sigma K^{(3)}(\mathbf{k}',\ \mathbf{k}'')\,\delta(\mathbf{k}' + \mathbf{k}'' - \pi\mathbf{b} - \mathbf{k}) \\ & \times [\Phi^+_{k'}\Psi^+_{k''}(\vartheta_k + \zeta_k) + \Phi_{k'}\Psi_{k''}(\vartheta^+_k + \zeta^+_k)] \\ & + \Sigma K^{(4)}(\mathbf{k})\,\delta(\mathbf{k}' - \mathbf{k}'' - \mathbf{k}) \\ & \times [(\zeta^+_{k'}\zeta_{k''} + \vartheta^+_{k'}\vartheta_{k''})(\vartheta_k - \zeta_k) \\ & + (\zeta_{k'}\zeta^+_{k''} + \vartheta_{k'}\vartheta^+_{k''})(\vartheta^+_k - \vartheta^+_{k'})] \end{aligned} \tag{3.4}$$

(summation over all indices). The form of the coefficients $K^{(i)}$ is not given here on account of its complexity.

4. ANALYSIS OF SEPARATE TERMS OF THIRD ORDER

The physical meaning of the separate terms of the expression (3.4) can be easily explained directly from the form of the operators. Thus, for example, the term $\Psi^+_{k'}\Psi_{k''}\vartheta_k$ corresponds to the annihilation of a left-hand exciton with quasi-momentum **k**. The energy of the exciton is thereby transferred to a hole with quasi-momentum **k**″, as a result of which the hole acquires a quasi-momentum **k**′. It must be mentioned that this process proceeds with conservation of momentum. The term associated with this one, $\Psi_{k'}\Psi^+_{k''}\vartheta^+_k$, corresponds to the inverse process, the process whereby a left-hand exciton with quasi-momentum k arises at the expense of the kinetic energy of a hole with quasi-momentum **k**′. The hole as a result of this process acquires a quasi-momentum **k**″. This process also takes place with conservation of momentum, as, by the way, do all the remaining processes described by the Hamiltonian (3.4).

Great interest attaches to the terms of the Hamiltonian (3.4) containing the operators $\Phi^+_{k'}\Psi^+_{k''}\xi_k$ and $\Phi^+_{k'}\Psi^+_{k''}\vartheta_k$. These operators describe the process of spontaneous annihilation of a right- or left-hand exciton with the quasi-momentum **k**, with simultaneous formation of a doublet with quasi-momentum **k**′ and a hole with quasi-momentum **k**″. The associated operators $\Phi_{k'}\Psi_{k''}\xi^+_k$ and $\Phi_{k'}\Psi_{k''}\vartheta^+_k$ correspond to the inverse processes --- processes of production of right-hand and left-hand excitons by recombination of a hole and a doublet. The physical meaning of the remaining operators is also clear from their individual forms.

The Hamiltonian $H_{(3)}$ can be regarded as the perturbation of a system, the stationary state of which is determined by the Hamiltonian $H_{(2)}$. In this way $H_{(3)}$ determines the transitions between the stationary states. The quantities $K^{(i)}$(**k**, **k**′, **k**″) determine the probabilities of the corresponding transitions. The computation of these probabilities will not be given in the present article; however, it is already clear from the Hamiltonian itself how, within the frame of the polar model of atomic semiconductors, one can proceed to the examination of very detailed processes.

5. SOME REMARKS ON THE HAMILTONIAN OF AN EXTERNAL PERTURBATION

Let $U(r, t)$ represent the operator of an external perturbation. We shall write down the Hamiltonian of an external perturbation in terms of the quantized wave functions

$$H' = \int \Psi^+(x)\, U(\mathbf{r},\ t)\, \Psi(x)\, dx.$$

Substituting the quantized wave functions into this and summing over spins we get

$$H' = \Sigma U_{q_1 i_1; q_2 i_2}\, a^+_{q_1 i_1 s}\, a_{q_2 i_2 s}\,, \tag{5.1}$$

where $U_{q_1 i_1; q_2 i_2}$ is the matrix element of the operator $U(r, t)$ computed by means of the atomic wave functions of the normal and first excited states, where $i = 0$ corresponds to the ground state and $i = 1$ to the excited state. a^+, a are the usual Fermi operators of the second quantization for creation and annihilation of the electron at the corresponding site.

Let us examine some of the possible transitions--in the first place, processes not connected with the generation of excitons. Obviously such processes are described by a Hamiltonian of the form

$$H_0' = \sum_{q_1 q_2 s} U_{q_1 0;\, q_2 0} [a^+_{q_1 0 s} a_{q_2 0 s} + a^+_{q_2 0 s} a_{q_1 0 s}]. \quad (5.2)$$

After changing to Bose operators we get in the quasi-classical approximation

$$H_0' = {}^1/_2 \sum_{qq'} U_{q0q'0} (\Phi^+_q \Phi_{q'} + \Phi^+_{q'} \Phi_q - \Psi^+_q \Psi_{q'} - \Psi^+_{q'} \Psi_q). \quad (5.3)$$

Therefore, as before, terms corresponding to the creation and annihilation of doublets and holes disappear in the quasi–classical approximation. From this we immediately obtain the following important result. If the external perturbation is a light wave, then holes and doublets cannot arise upon absorption of the light, i.e., conduction states cannot appear. Consequently, the absorption of light in atomic semiconductors, from the point of view of the proposed theory, is photoelectrically inactive and bears a purely exciton character. If one gives up the quasi-classical approximation, the creation of holes and doublets will be possible; however, the probability of this process will be very small in comparison with the probability of exciton absorption.

As regards processes described by the Hamiltonian (5.3) in the case when the perturbation is a light wave, this Hamiltonian then, as it is not difficult to convince oneself, describes the Compton scattering of light by free doublets and holes. In what follows we shall not be interested in this process.

In order to study the exciton mechanism for absorption of light, it is necessary to put $q_i = q_2$, $i_2 = 0$, $i_1 = 1$ into (5.1); then we obtain

$$H_{01}' = \sum_{qs} [U_{q1q0}\, a^+_{q1s} a_{q0s} + U_{q0q1}\, a^+_{q0s}\, a_{q1s}]. \quad (5.4)$$

Since the matrix element U_{q1q0} is computed by use of real functions, $U_{q0q1} = U_{q1q0}$. Going over to Bose operators and to the quasi-classical approximation, we get

$$H_{01}' = 2^{-1/2} \sum_q U_{q1q0} (\alpha^+_q + \alpha_q) - 2^{-1/2} \sum_q U_{q1q0} (\beta^+_q + \beta_q). \quad (5.5)$$

In the Hamiltonian (5.1) are contained terms describing even more complicated processes, for example, the simultaneous absorption of light with the excitation of an exciton and scattering by holes, etc. However, all these processes have a considerably smaller probability, and in the sequel we shall not consider them either. Here we only remark that from this theory arises the possibility of processes, whereby a light wave causes the annihilation of excitons already existing in the crystal with formation of holes and doublets, which leads to an additional conductivity of the crystal.

If the external perturbation is a static one, for example, the force field of an impurity atom, the latter can also cause a series of transitions between states with the same initial and final energy, for example, the decay of excitons with formation of holes and electrons, etc. The corresponding Hamiltonian can also be obtained from (5.1).

Going over to the Bose operators (and to the quasi-classical approximation), we obtain, after the corresponding transformations,

$$\begin{aligned} H_{01}' = {} & 2^{-1/2} \Sigma U_{q1q0} (\alpha_q + \alpha^+_q - \beta^+_q - \beta_q) \qquad (5.6) \\ & + {}^1/_2 \Sigma U_{f0q0} (\Phi^+_q \Phi_f + \Phi^+_f \Phi_q - \Psi^+_q \Psi_f - \Psi^+_f \Psi_q) \\ & + 2^{-1/2} \Sigma U_{f1q0} [(\beta_f - \alpha_f) \Psi^+_f \Psi_q + (\beta^+_f - \alpha^+_f) \Psi_f \Psi_q] \\ & + 2^{-1/2} \Sigma U_{f1q0} [(\beta_f + \alpha_f) \Psi^+_f \Phi^+_q + (\beta^+_f + \alpha^+_f) \Psi_f \Phi_q]. \end{aligned}$$

After the transition to the space of quasi-momenta, it is easy to explain the meaning of each term of the Hamiltonian (5.6). Thus for example, the first term of the Hamiltonian gives those transitions in which the law of conservation of energy cannot be satisfied (they can be considered as virtual transitions). Therefore, in considering processes of interaction and transformation of quasi-particles into each other this term can be discarded. The second term of the Hamiltonian (5.6) describes the scattering of holes and doublets from the impurity site. The quasi-momentum of the doublet or hole in such a process takes on a different value after scattering.

The third term of (5.6) describes the process of annihilation of a right-hand or left-hand exciton at the impurities with transfer of their energy already existing in the crystal to the hole, and the inverse process of the creation of a right- or left-hand exciton at the expense of the kinetic energy of a hole. The analogous process with doublets is impossible, since in this case a two-electron transition would be necessary, which cannot be produced by a static field. From this it follows that that part of the mobility which is caused by the presence of impurities will be different for holes and doublets in exciton semiconductors.

Finally, the fourth term of the Hamiltonian (5.6) describes the process of decay of a right- or left-hand exciton at an impurity site into a doublet and a hole, and the inverse process of formation of a right- or left-hand exciton at the expense of recombination of a hole and a doublet at the impurity site.

6. DERIVATION OF AN EXPRESSION FOR THE PROBABILITY OF DECAY OF AN EXCITON AT AN IMPURITY WITH FORMATION OF A DOUBLET AND A HOLE

Let us consider the decay of an exciton (for example, a left-hand one) at the site of an impurity. We locate the origin of coordinates at this site. From (5.6) it follows that the energy operator of the perturbation K has the form

$$K=2^{-1/2}\Sigma\, U_{q_1q'0}\,\alpha_q \Psi_q^+\Phi_{q'}^+ . \tag{6.1}$$

Let $U(\mathbf{r})$ be the perturbing potential of the impurity; then

$$U_{q1q'0}=\int\varphi_1^*(\mathbf{r}-\mathbf{R}_q)\,U(\mathbf{r})\,\varphi_0(\mathbf{r}-\mathbf{R}_{q'})\,d\tau. \tag{6.2}$$

Substituting (6.2) into (6.1) and going over to momentum space we get

$$K=((2\pi)^3 N^{-1/2}/\sqrt{2V})\sum_{\alpha\beta\gamma}\vartheta_\alpha\Psi_\beta^+\Phi_\gamma^+ \tag{6.3}$$
$$\times U(|\mathbf{k}_\gamma+\mathbf{k}_\beta-\mathbf{k}_\alpha-\pi\mathbf{b}|)$$
$$\times\varphi_1(\mathbf{k}_\beta-\mathbf{k}_\alpha-\pi\mathbf{b})\,\varphi_0(\mathbf{k}_\gamma),$$

where φ_1 and φ_2 are the atomic functions in momentum space, and U is the Fourier component of the potential, determined by the expression

$$U(|\mathbf{k}_\gamma+\mathbf{k}_\beta-\mathbf{k}_\alpha-\pi\mathbf{b}|) \tag{6.4}$$
$$=1/V_0\int U(|\mathbf{r}|)\,e^{i(\mathbf{k}_\gamma+\mathbf{k}_\beta-\mathbf{k}_\alpha-\pi\mathbf{b})\mathbf{r}}\,d\tau,$$

where V_0 is the volume associated with one atom.

The transition probability can be computed according to the familiar formula

$$w=(2\pi/h)\int|K_{\lambda\mu}|^2\,\delta(E_\lambda-E_\mu)\,d\mu, \tag{6.5}$$

where $K_{\lambda\mu}$ is the matrix element of the operator (6.3) for the transition between states $\lambda\to\mu$ (exciton → doublet plus hole), and the integration is carried out over all possible finite states. We note that in the case at hand, the law of conservation of momentum is not satisfied, since the decay of the exciton takes place in the force field of the impurity. The law of conservation of energy is taken into account by the δ-function in Eq. (6.5).

Calculating the values of the matrix elements of the operator $\vartheta_\alpha\Psi_\beta^+\Phi_\gamma^+$, stipulating that there is one exciton present and paying attention to the fact that the average occupation number of holes and doublets is ≪ 1, we get from Eq. (6.5)

$$w=(\pi/\hbar N)\int|\varphi_1(\mathbf{k}_\beta-\mathbf{k}_\alpha-\pi\mathbf{b})|^2\,|\varphi_0(\mathbf{k}_\gamma)|^2 \tag{6.6}$$
$$\times U^2(|\mathbf{k}_\gamma+\mathbf{k}_\beta-\mathbf{k}_\alpha-\pi\mathbf{b}|)$$
$$\times\delta(E_e-\varepsilon_\beta-\varepsilon_\gamma)\,d\mathbf{k}_\beta d\mathbf{k}_\gamma,$$

where E_e is the exciton energy, ϵ_β and ϵ_γ, respectively, the energy of the hole and the doublet. Changing to spherical coordinates in this expression, and carrying out the integration over angles, we get

$$w=\frac{4\pi^3\mu_d}{\hbar^3 N\varkappa}\int_0^\alpha k_\beta\,|\varphi_0(k'_\gamma)|^2 \tag{6.7}$$
$$\times\int_{|k_\beta-\varkappa|}^{k_\beta+\varkappa}|\varphi_1(y)|^2\int_{|k'_\gamma-y|}^{k'_\gamma+y} zU^2(z)\,dz\,dy\,dk_\beta,$$

$$k'_\gamma=\sqrt{\alpha^2-k_\beta^2};$$

$$\alpha=\sqrt{(2\mu_d/h^2)(h\nu-A)};\quad \varkappa=|\mathbf{k}_\alpha+\pi b|.$$

In Eq. (6.7) $\varphi_1(y)$ represents only the radial part of the wave functions; an averaging has been carried out over the angular part.

If there are few sites of the impurity in the crystal, then the action of each on the excitons will be the same as though there were no other impurity sites. The probability of decay of an exciton in such a crystal into a doublet and a hole will be equal to $w=N_{\text{imp}}W$, where N_{imp} is the number of impurity sites in the crystal. From (6.7) it follows that $W\approx 1/N$, therefore, $w\approx N_{\text{imp}}/N$. In this way the probability of decay of an exciton in the crystal into a doublet and a hole is directly proportional to the concentration of impurities. From this it follows that the photoconductivity of an atomic semiconductor, caused by an impurity, is directly proportional to the concentration of impurities. If the probability of spontaneous decay of an exciton is small, then we shall have principally impurity photoconductivity.

The effective cross section can be computed by the formula given by Sokolov and Ivanenko[4]:

$$\sigma = wV / v_e, \tag{6.8}$$

where v_e is the velocity of the exciton. Using the law of conservation of momentum, we obtain

$$v_e = \Delta E_e / \mu_e c. \tag{6.9}$$

Consequently,

$$\sigma = \frac{4\pi^3 \mu_d \mu_e cV}{\hbar^3 \varkappa \Delta E_e N} \tag{6.10}$$

$$\times \int_0^\alpha k_\beta |\varphi_0 (k'_\gamma)|^2 \int_{|k_\beta - \varkappa|}^{k_\beta + \varkappa} |\varphi_1 (y)|^2 \int_{|k'_\gamma - y|}^{k'_\gamma + y} zU^2 (z) \, dz \, dy \, dk_\beta.$$

As an example of the application of Eq. (6.10), we consider the decay of an exciton at an empty site. In this case the perturbing potential can be set equal to the potential of an isolated atom (with the opposite sign). Therefore, we put

$$U(r) = \frac{ze^2}{r} - e^2 \int \frac{\rho (\mathbf{r}'') \, dv''}{|\mathbf{r}'' - \mathbf{r}|}, \tag{6.11}$$

where $\rho(\mathbf{r})$ is the density of the distribution of electrons. Let us take

$$\rho(\mathbf{r}) = (z\gamma^3 / 8\pi) e^{-\gamma r}. \tag{6.12}$$

For the rest of the calculation it is necessary to choose definite wave functions. For the evaluation we choose for φ_0 the wave function of a hydrogen atom in the 2*s* state, and for φ_1 that of the 3*p* state[5]. The wave function φ_1 was averaged over angles in the following way. Since *p*-states are triply degenerate, φ_1^2 can be set equal to

$$|\varphi_1|^2 = \tfrac{1}{3} (|\varphi_1^{(1)}|^2 + |\varphi_1^{(2)}|^2 + |\varphi_1^{(3)}|^2), \tag{6.13}$$

where $\varphi_1^1, \varphi_1^2, \varphi_1^3$ are the three actual functions for the *p*-state.

Substituting the wave functions and (6.11) into (6.10) we obtain

$$\sigma = \frac{2^{20} 3^5 \pi^2 c \mu_d \mu_e e^4 z^2 n}{\hbar^3 \Delta E_e \lambda_s^3 \lambda_p^5} \frac{\alpha^4 (\alpha^2 + \varkappa^2)}{(\alpha^2 + \gamma^2 + \varkappa^2)^2}. \tag{6.14}$$

Determination of the numerical value shows that σ changes within unusually wide limits in its dependence on the choice of the effective charges for the *s*- and *p*-states. The results of such an evaluation, as the parameters λ_p and λ_s are changed, are as follows:

λ_s	10^9	10^9	5×10^8	5×10^8
λ_p	3×10^9	2×10^9	2×10^9	3×10^9
σ(cm^2)	4×10^{-16}	3×10^{-15}	2.5×10^{-14}	3×10^{-15}

This shows that σ is very sensitive to the behavior of the wave functions and that for an actual calculation of σ it is necessary to know the exact atomic wave functions in momentum space.

The existing methods for computing atomic functions (the Hartley-Fock method, the variational method) allow one to compute the atomic functions in configuration space. The analogous problem in momentum space is still not solved; therefore, we cannot exhibit numerical values for σ for concrete cases. However, the result obtained shows that the exciton can decay at an impurity with formation of charge carriers even in the case when the site is empty, i.e., does not contain bound charges in it.

7. KINETICS OF PHOTOCONDUCTIVITY

As was explained in Secs. 5 and 6, the third-order terms in the Hamiltonian of the polar model of an atomic semiconductor determine the probabilities of different processes of transformation of certain quasi-particles into others. Furthermore, the theory permits us to calculate the probability of formation of excitons under the action of incident light. This gives the probability, knowing the matrix elements of the corresponding transitions, of writing down the kinetic equations[6], but in the present work the interaction of quasi-particles with phonons will not be considered. Moreover, this interaction is essential for the study of kinetics in semiconductors, since without consideration of the recombination of holes and doublets with radiation of phonons it is impossible to obtain reasonable equations of kinetics, leading to a correct value of the number of quasi-particles in stationary states. Therefore, the equations of macrokineties for holes and doublets are supplemented by phenomenological terms taking into account the thermal recombination of holes and doublets.

It is postulated that the energy of excitation of excitons is much larger than the excitation energy of holes and doublets; thus in the first approximation it is possible to assume that, in contrast with doublets and holes, thermal excitons do not exist.

The equations of macrokinetics have the form

$$dM'/dt = \lambda Q - 2\gamma M_0 M', \tag{7.1}$$

$$dL'/dt = \lambda Q - 2\gamma M_0 L',$$

$$dQ/dt = 2w_0 n_\alpha + (w_0 + \lambda)Q,$$

where M' is the number of L', photo-holes of photo-doublets, Q of excitons (right- and left-hand), M_0 of thermal holes, γ the recombination coefficient of holes and doublets, n_α the number of photons, w_0 the absorption coefficient of a photon, λ the coefficient of the probability of decay of an exciton into a doublet and a hole (at an impurity atom or spontaneously).

Solving the system of Eqs. (7.1) we obtain

$$M' = M'_{\text{stat}}\left[1 + \frac{e^{-(w_0+\lambda)t}}{(w_0+\lambda)/2\gamma M_0 - 1} + \frac{e^{-2\gamma M_0 t}}{2\gamma M_0/(w_0+\lambda) - 1}\right], \tag{7.2}$$

$$L' = M',\quad Q = Q_{\text{stat}}[1 - e^{-(w_0+\lambda)t}],$$

$$M'_{\text{stat}} = w_0 n_\alpha \lambda / \gamma M_0 (w_0 + \lambda); \tag{7.3}$$

$$Q_{\text{stat}} = 2w_0 n_\alpha / (w_0 + \lambda).$$

The dependence on temperature enters only into M_0, $M_0 \sim e^{-\Delta E/2kT}$, where ΔE is the energy gap between the currentless states of the crystal and the conduction band. From (7.2) it follows that the number of excitons in the crystal, as one was led to expect, does not depend on temperature. As rerards the kinetics of photo-holes, the following limiting cases can occur:

a) $w_0 + \lambda \gg 2\gamma M_0$, (7.4)

then $M' = M'_{\text{stat}}(1 - e^{-2\gamma M_0 t})$,

b) $w_0 + \lambda \ll 2\gamma M_0$, (7.5)

then $M' = M'_{\text{stat}}(1 - e^{-(w_0+\lambda)t})$.

With increasing temperature the number of thermal holes M_0 increases and, consequently, the thermal conductivity increases, while the stationary value of the photoconductivity will decrease with increase of temperature,

$$\Delta\sigma_{\text{stat}} \sim e^{\Delta E/2kT}. \tag{7.6}$$

This results from the fact that with the increase of temperature the recombination of photo-holes with doublets (and photo-doublets with holes) proceeds more intensively.

As was shown by Zhuze and Ryvkin[7], the stationary concentration of carriers of photocurrent can be represented in the form of a product of a set of parameters

$$\Delta n_{\text{stat}} = \tau k \beta I, \tag{7.7}$$

where τ is the lifetime of the carriers of photocurrent, k the coefficient of absorption, I the intensity of the light, and β the coefficient of the photoeffect which, if I is measured by the number of quanta falling per second on a unit surface area, has the sense of the "quantum yield".

The photoconductivity is determined by the formula

$$\Delta\sigma_{\text{stat}} = eu\Delta u_{\text{stat}}, \tag{7.8}$$

where e is the electronic charge, u is the mobility of the carriers of photocurrent. The number of quanta absorbed per second in a unit volume is equal to $kI = 2w_0 n_\alpha$; therefore,

$$\Delta n_{\text{stat}} = M'_{\text{stat}} = 2\beta\tau w_0 n_\alpha. \tag{7.9}$$

From Eqs. (7.3) and (7.9) we obtain for the quantum yield

$$\beta = \lambda / 2\gamma\tau M_0 (w_0 + \lambda). \tag{7.10}$$

Let us consider two limiting cases:

a) $w_0 + \lambda \gg 2\gamma M_0$, then from (7.4) and (7.10) we have

$$\tau = 1/(2\gamma M_0),\quad \beta = \lambda/(w_0 + \lambda);$$

b) $w_0 + \lambda \ll 2\gamma M_0$, then from (7.5) and (7.10) we have

$$\tau = 1/(w_0 + \lambda),\quad \beta = \lambda/2\gamma M_0.$$

Let us now write out the temperature dependence of $\Delta\sigma_{\text{stat}}$, τ, and β in both limiting cases:

$$\text{a) } \Delta\sigma_{\text{stat}} \sim e^{\Delta E/2kT}, \ \tau \sim e^{\Delta E/2kT}, \ \beta \sim \text{const};$$
$$\text{b) } \Delta\sigma_{\text{stat}} \sim e^{\Delta E/2kT}, \ \tau \sim \text{const}, \ \beta \sim e^{\Delta E/2kT}$$

All the reasoning carried out up to this time is valid for sufficiently high temperatures ($M' \ll M_0$). As regards the region of low temperatures, here it is already impossible to assume $M' \ll M_0$. This introduces changes into the kinetic equations. In particular, the first of the Eqs. (7.1) ceases to be linear and takes the form

$$dM'/dt = \lambda Q - 2\gamma M_0 M' - \gamma M'^2. \qquad (7.11)$$

The solution of this equation can be expressed in Bessel functions of order

$$p = \frac{2}{w_0 + \lambda}\left[\gamma^2 M_0^2 + 2\gamma\,\frac{w_0 n_\alpha + \lambda}{w_0 + \lambda}\right]^{1/2}.$$

Naturally it is difficult to say anything definite about the behavior of the solution; therefore, in the case of low temperatures, we have limited ourselves to finding the stationary value of M'. In this case

$$M'_{\text{stat}} = [2w_0 n_\alpha \lambda / \gamma(w_0 + \lambda)]^{1/2} = \text{const}. \qquad (7.12)$$

Since $M'_{\text{stat}} \sim \Delta\sigma_{\text{stat}}$, we obtained the result that at low temperatures the stationary photoconductivity does not depend on the temperature.

For the limiting case b) we know no experimental data supporting the temperature variation of τ and β in this case. It is possible that this case is not realized.

If one compares the temperature variation of $\Delta\sigma_{\text{stat}}$, τ and β in case a) with experimental data for copper oxide, obtained by Zhuze and Ryvkin[8], in the region of sufficiently high temperatures ($T > -40°$) one gets complete agreement with the temperature variation of $\Delta\sigma_{\text{stat}}$, τ and β obtained experimentally. As regards the region of low temperatures, the temperature variation of $\Delta\sigma_{\text{stat}}$ agrees well with the experimental data. To say anything definite about the temperature variation of the lifetime τ and the quantum yield β in the case of low temperatures is hardly possible; but since the predictions of the present theory differ from those of the theory of Zhuze and Ryvkin, it is hardly possible to obtain the same temperature dependence for τ and β as obtained by them. In the first place, these authors consider that the photoconductivity remains linear even at low temperatures; in the second place, they consider that the decay of an exciton takes place at defects, in which are found electrons removed from the normal band, whereby upon the decay of an exciton the electron is thrown back from the defect to the conduction band[9]. Thus the number of "effective" decay centers depends on the temperature, and their states does not change upon the decay of an exciton (empty sites). Apparently this difference must lead to another temperature dependence of the quantum yield in the region of low temperatures.

We must also mention that copper oxide does not belong to the atomic semiconductors which are considered in the present theory; however, it is entirely possible that the dependences proposed by Zhuze and Ryvkin are of general character.

Therefore, at sufficiently high temperatures, the expression for the stationary photoconductivity created by holes has the form

$$\Delta\sigma_{\text{stat}} = \lambda e u k I / 2\gamma M_0 (w_0 + \lambda). \qquad (7.13)$$

At low temperatures the formula

$$\Delta\sigma_{\text{stat}} = (\lambda k I)^{1/2} e u_p / \gamma^{1/2} (w_0 + \lambda)^{1/2} \qquad (7.14)$$

holds.

Analogous expressions can also be written down for the photoconductivity caused by doublets. However, the hole photocurrent can have a different value from that of the doublet photocurrent, if there is a different mobility for doublets and for holes.

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On the Theory of the Stripping Reaction

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This investigation presents a simple derivation of the Butler formula for the angular distribution in the (*d, p*) reaction. The scattering of the deuteron and proton waves in the nuclear field is included. The emamined theory leads to Serber's results in the limiting case of large energies for the impinging deuterons.

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1. AS is known, the stripping reaction is an important means for investigating nuclear properties. From the angular distribution of the products of the (*d, p*) and (*d, n*) reactions it is possible to determine the spin and parity of the corresponding state of the residual nucleus, if the spin and parity of the ground state of the initial nucleus are known. The stripping reaction theory for bombarding deuterons of moderate energies (~10 mev) was first presented by Butler[1], who determined the angular distribution of stripping reaction products by using the continuity condition of the wave functions on the nuclear surface. Since Butler's derivation is extremely complicated, there are several other investigations[2-6] in which the same results are obtained by different means. In particular, Bhatia[2] made use of the Born approximation, which led to correct results, though there is little justification for the application of such an approximation to the energy region mentioned. Incomplete agreement of the theoretical data with the results of experiments indicates the importance of considering the nuclear and Coulombic scattering of the particles participating in the reaction. This was done in some of the investigations[5-7]. It may be that an important role is also played by interference between the stripping reaction and the processes of compound nucleus formation.

In this article, using the method developed by Landau and Lifshitz[8], we determine the angular distribution of protons in the (*d, p*) reaction, and calculate the scattering of deuteron and proton waves in the nuclear field. Calculation of the scattering reveals the presence of partial polarization of the freed protons. If a neutron is captured by the nucleus on the virtual level, the stripping reaction cross section is proportional to the level width. When the impinging deuterons have enough energy, the total cross section of the stripping reaction is identical with the cross section predicted by Serber[9]. In this case the distribution in angles and energy of the freed protons corresponds to that of the "transparent model" in the Serber theory[9].

2. Schrödinger's equation, which describes the motion of a neutron + proton in a field due to the presence of a nucleus, may be written as

$$\{T_n+T_p+V_n+V_p+V_{np}-E\}\Psi=0, \qquad (1)$$

where T_n and T_p are the kinetic energy operators of the neutron and proton, V_n and V_p are the interaction potentials of neutron and proton with the nucleus, V_{np} is the nuclear interaction potential of a neutron with a proton, and E is the total energy in the system.

To solve Eq. (1), we expand the unknown function Ψ in wave functions of the proton released by the deuteron disintegration. These wave functions, which we shall designate as $\psi_{\mathbf{k}_p}$ ($\mathbf{k}_p$ being the wave vector of the outgoing proton), satisfy the equation

$$\{T_p+V_p-E_p\}\psi_{\mathbf{k}_p}=0 \qquad (2)$$

($E_p=\hbar^2k_p^2/2M$ is the proton energy), with $\psi_{\mathbf{k}_p}$ being composed of a plane wave and converging spherical wave at infinity. Let us assume that the function $\psi_{\mathbf{k}_p}$ is subject to the following normalization condition:

$$\int \psi_{\mathbf{k}_p}\psi^*_{\mathbf{k}'_p}\,d\mathbf{r}_p=\delta_{\mathbf{k}_p\mathbf{k}'_p}. \qquad (3)$$

The solution of Eq. (1) can be represented as

$$\Psi(\mathbf{r}_n,\mathbf{r}_p)=\sum_{\mathbf{k}_p} a(\mathbf{r}_n,\mathbf{k}_p)\,\psi_{\mathbf{k}_p}(\mathbf{r}_p), \qquad (4)$$

where $a(\mathbf{r}_n,\mathbf{k}_p)$ are certain functions of the coordinates of the neutron and wave vector of the outgoing proton. Obviously, $a(\mathbf{r}_n, \mathbf{k}_p)$ is the wave function of the neutron formed by the disintegration of a deuteron and corresponds to a proton with wave vector $\mathbf{k}_p$.

Substituting (4) in (1), and making use of the orthogonality of the $\psi_{\mathbf{k}_p}$ functions in (3), we obtain the following equations for the functions $a(\mathbf{r}_n, \mathbf{k}_p)$:

$$\{\Delta_n + (2M/\hbar^2)(E - E_p - V_n)\} \tag{5}$$

$$\times\, a(\mathbf{r}_n, \mathbf{k}_p) = f(\mathbf{r}_n),$$

$$f(\mathbf{r}_n) = (2M/\hbar^2)\int \psi^*_{\mathbf{k}_p}(\mathbf{r}_p)\, V_{np}\, \Psi(\mathbf{r}_n, \mathbf{r}_p)\, d\mathbf{r}_p. \tag{6}$$

Equation (5) is exact. To obtain an approximate solution of this equation we substitute the approximate function Ψ_0 for Ψ, where

$$\Psi_0(\mathbf{r}_n, \mathbf{r}_p) = \varphi(r)\,\psi_{\mathbf{k}d}(\mathbf{r}_d), \tag{7}$$

$\varphi(r)$ is the wave function of the ground state of the deuteron and $\psi_{\mathbf{k}d}(\mathbf{r}_d)$ is the wave function of the deuteron moving as a unit in the nuclear field. At infinity, the function $\psi_{\mathbf{k}d}(\mathbf{r}_d)$ is composed of the impinging plane wave and a scattered diverging spherical wave.

Because of the short range of the nuclear forces, one can use the relation[8]

$$V_{np}\varphi(r) = -(4\pi\hbar^2/M)\sqrt{\alpha/2\pi}\,\delta(\mathbf{r}) \tag{8}$$

in the integral in (6). In this way we obtain

$$f(\mathbf{r}_n) = -8\pi\sqrt{\alpha/2\pi}\,\psi^*_{\mathbf{k}_p}(\mathbf{r}_n)\,\psi_{\mathbf{k}_d}(\mathbf{r}_n). \tag{6$'$}$$

Henceforth, we shall disregard the possibility of the occurrence of a deuteron as a unit inside the nucleus, and therefore $f(\mathbf{r}_n)$ will differ from zero only when $r_n > R$ ($R \approx R_0 + R_d$), where R_0 is the radius of the nucleus and R_d the radius of the deuteron.

Expanding $a(\mathbf{r}_n, \mathbf{k}_p)$ in spherical functions

$$a(\mathbf{r}_n, \mathbf{k}_p) = \sum_{l,m} a_{lm}(r_n)\, Y_{lm}(\vartheta_n, \varphi_n), \tag{9}$$

we obtain the following equations for the coefficients $a_{lm}(r_n)$:

$$\left\{\frac{d^2}{dr_n^2} + \frac{2}{r_n}\frac{d}{dr_n} - \frac{l(l+1)}{r_n^2} \right. \tag{10}$$

$$\left. + \frac{2M}{\hbar^2}(E - E_p - V_n)\right\} a_{lm}(r_n) = f_{lm}(r_n)$$

$$f_{lm}(r_n) = \int Y^*_{lm}(\vartheta_n, \varphi_n)\, f(\mathbf{r}_n)\, do_n. \tag{11}$$

The required solution for Eq. (10) must be finite as $r_n \to 0$ and must represent a diverging spherical wave when $r_n \to \infty$.

Let $R_l^{(1)}$ be the solution of Eq. (10) with the right side zero which satisfies the bounded condition for $r_n \to 0$. In the region beyond the action of nuclear forces ($r_n > R$) this solution is clearly of the form

$$R_l^{(1)} = j_l(k_n r_n) + b h_l^{(1)}(k_n r_n), \tag{12}$$

$$r_n > R,\ k_n = \sqrt{(2M/\hbar^2)(E - E_p)},$$

where b is a coefficient determined by the conditions of continuity and boundedness of the solution. Let $R_l^{(2)}$ designate the second independent homogeneous solution of Eq. (10), which represents the divergent spherical wave at infinity. In the outside region this solution has the form

$$R_l^{(2)} = h_l^{(1)}(k_n r_n),\ r_n > R. \tag{13}$$

Knowing $R_l^{(1)}$ and $R_l^{(2)}$, one can construct a solution of the inhomogeneous equation [i.e., Eq. (10)] which satisfies the given boundary conditions, viz.,

$$a_{lm}(r_n) = A R_l^{(2)}(r_n)\int_0^{r_n} R_l^{(1)}(r)\, f_{lm}(r)\, r^2 dr \tag{14}$$

$$+ A R_l^{(1)}(r_n)\int_{r_n}^{\infty} R_l^{(2)}(r)\, f_{lm}(r)\, r^2 dr,$$

where the coefficient A is determined from the condition

$$r^2\{R_l^{(1)} R_l^{(2)\prime} - R_l^{(1)\prime} R_l^{(2)}\} = 1/A. \tag{15}$$

Substituting solutions (12) and (13) into (15), we find that $A = -ik_n$.

Since $f_{lm}(r) = 0$ if $r < R$, then with the use of Eqs. (11) and (6), we finally obtain a solution for points $r_n < R$; thus,

$$a_{lm}(r_n) = i8\pi\sqrt{\alpha/2\pi}\, k_n R_l^{(1)}(r_n)\, I_l^m, \quad r_n < R, \tag{16}$$

$$I_l^m = \int_{r>R} h_l^{(1)}(k_n r)\, Y^*_{lm}(\vartheta, \varphi)\, \psi^*_{\mathbf{k}_p}(\mathbf{r})\, \psi_{\mathbf{k}_d}(\mathbf{r})\, d\mathbf{r}. \tag{17}$$

3. We now develop the general formula for the (d, p) stripping reaction cross section. For this purpose we compute the neutron flux S through the nuclear surface in the normal direction,

$$S = \frac{i\hbar}{2M} r_n^2 \int \left\{ a^* (\mathbf{r}_n, \mathbf{k}_p) \frac{\partial a(\mathbf{r}_n, \mathbf{k}_p)}{\partial r_n} \right. \tag{18}$$

$$\left. - a(\mathbf{r}_n \mathbf{k}_p) \frac{\partial a^* (\mathbf{r}_n, \mathbf{k}_p)}{\partial r_n} \right\} do_n \Big|_{r_n = R}.$$

Substituting $a(\mathbf{r}_n, \mathbf{k}_p)$ as given by (9) and using (16), we obtain

$$S = 32\pi\alpha\, |k_n|^2 \tag{19}$$

$$\times \sum_l \frac{i\hbar}{2M} R^2 \left\{ R_l^{(1)*} \frac{dR_l^{(1)}}{dr_n} - R_l^{(1)} \frac{dR_l^{(1)*}}{dr_n} \right\}_{r_n = R} \sum_m |I_l^m|^2.$$

Since the proton wave function $\psi_{\mathbf{k}_p}$ is normalized to a δ-function in wave vector space, then by multiplying S by the element of solid angle do_p we obtain the number of neutrons absorbed per unit of time that are associated with the protons emitted within do_p. Dividing this figure into the flux density of impinging deuterons, which is $\hbar k_d/2M$, we obtain the following equation for the stripping reaction cross section:

$$d\sigma = 32\pi \frac{\alpha}{k_d} |k_n|^2 R^2 \tag{20}$$

$$\times \sum_l i \left\{ R_l^{(1)*} \frac{dR_l^{(1)}}{dR} - R_l^{(1)} \frac{dR_l^{(1)*}}{dR} \right\} \sum_m |I_l^m|^2 do_p.$$

Thus, the problem of determining the angular distribution of the freed protons is reduced to the computation of the integral, Eq. (17), in which the deuteron and proton wave functions are given by

$$\psi_{\mathbf{k}_d}(\mathbf{r}) = e^{i\mathbf{k}_d \mathbf{r}} + 2\pi \tag{21}$$

$$\times \sum_{l,m} i^l (\beta_l^d - 1)\, h_l^{(1)}(k_d r)\, Y_{lm}^* (\vartheta_{k_d}, \varphi_{k_d})\, Y_{lm}(\vartheta, \varphi),$$

$$\psi_{\mathbf{k}_p}(\mathbf{r}) = (2\pi)^{-\frac{3}{2}} \left\{ e^{i\mathbf{k}_p \mathbf{r}} + 2\pi \right. \tag{22}$$

$$\left. \times \sum_{l,m} i^l (\beta_l^{p*} - 1)\, h_l^{(2)}(k_p r)\, Y_{lm}^* (\vartheta_{k_p}, \varphi_{kp})\, Y_{lm}(\vartheta, \varphi) \right\}.$$

Complex coefficients β_l^d and β_l^p characterize the scattering of deuteron and proton waves in the nuclear field[11]

Inclusion of the scattering of deuteron and proton waves leads to partial polarization of the freed protons*, which is fully determined by the integrals I_l^m, that is,

$$P = \mp \frac{2}{3(2j+1)} \left(\sum_m m\, |I_l^m|^2 \Big/ \sum_m |I_l^m|^2 \right), \tag{23}$$

$$j = l \pm {}^1/_2.$$

Far from resonance, the deuteron and proton experience almost complete reflection at the edge of the nucleus, and consequently, nuclear wave scattering can be regarded approximately as scattering by an impenetrable sphere of radius R[13]. For this case coefficients β_l^d and β_l^p (if we disregard Coulomb interaction) are determined by the equation[13]

$$\beta_l^{d,p} = 1 - 2j_l(k_{d,p}R)\,[j_l(k_{d,p}R) \tag{24}$$

$$- i n_l(k_{d,p}R)] / [j_l^2(k_{d,p}R) + n_l^2(k_{d,p}R)].$$

Near energy resonance, resonance scattering plays a major role, in which case

$$\beta_l^{d,p} = 1 - i\Gamma_l^{d,p} \Big/ \left(E - E_l + \frac{i}{2}\Gamma_l \right), \tag{25}$$

where E_l is the resonance energy, Γ_l the total width of the resonance level, and Γ_l^d and Γ_l^p are partial widths. However, near resonance the deuteron penetrates into the nucleus with appreciable probability, and therefore, compound nucleus formation will play an important part in the (d, p) process. Interference between the two processes will lead to considerable change in the angular distribution of the reaction products.

In the limiting case of high energies, potential scattering, Eq. (24), becomes diffraction scattering, in which case we may set approximately[11]

$$\beta_l^{d,p} = \begin{cases} 0 & l \leqslant k_{l,p} R \\ 1 & l > k_{d,p} R. \end{cases} \tag{26}$$

However, even in the simplest cases, i.e., Eqs. (24) and (26), the integration of Eq. (17) is feasible only numerically. (In this connection, compare Ref. 14.) However, the integral can be easily calculated if deuteron and proton wave scattering in the nuclear field is disregarded.

$$I_l^m = \sqrt{\frac{2}{\pi}}\, i^l \frac{R^2}{k_n^2 - k^2} \left\{ j_l(kR) \frac{dh_l^{(1)}(k_n R)}{dR} \right. \tag{27}$$

$$\left. - h_l^{(1)}(k_n R) \frac{dj_l(kR)}{dR} \right\} Y_{lm}^* (\vartheta_k, \varphi_k),$$

$$k = |\mathbf{k}_d - \mathbf{k}_p|.$$

For this case we obtain Butler's result for the angular distribution of the protons, viz.,

$$d\sigma = \frac{16\alpha R^6 |k_n|^2}{\pi k_d [(\mathbf{k}_d - \mathbf{k}_p)^2 - k_n^2]^2} \tag{28}$$

$$\times \sum_l (2l+1)\, i \left\{ R_l^{(1)*} \frac{dR_l^{(1)}}{dR} - R_l^{(1)} \frac{dR_l^{(1)*}}{dR} \right\}$$

$$\times \left| j_l(kR) \frac{dh_l^{(1)}(k_n R)}{dR} - h_l^{(1)}(k_n R) \frac{dj_l(kR)}{dR} \right|^2 do_p.$$

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* The proton polarization caused by proton wave scattering has been investigated by Horowitz and Messiah[12].

Inclusion of the spins causes an additional factor to appear in Eq. (28) in the sum, viz.,

$$2(2j+1)/3(2i+1)(2l+1),$$

where i and j are the spins of the initial and final nucleus and l is the orbital moment transferred by the neutron to the nucleus.

In deriving Eq. (28) we neglected the possibility of the occurrence of a deuteron as a unit inside the nucleus. This assumption is justified for deuterons with impact parameters in excess of the effective nuclear radius R; therefore, the angular distribution found for the products of the stripping reaction will be quite exact for small angles.

4. If the energy of the impinging deuteron is great enough ($E_d \gg \epsilon$), a neutron will be captured by the nucleus on the virtual level (k_n real). In this case it is convenient to express the cross section Eq. (28) in terms of the partial width Γ_l^n which corresponds to an emission of a neutron with an orbital moment of l by the final nucleus. Obviously, the probability of neutron emission per unit time, $\Gamma_l^n/\hbar$, is equal to the neutron flux through the surface of a sphere with radius R in the normal direction[13], i.e.,

$$\Gamma_l^n/\hbar = -(i\hbar/2M)R^2 \times \{R_l^{(1)*}dR_l^{(1)}/dR - R_l^{(1)}dR_l^{(1)*}/dR\} \tag{29}$$

($R_l^{(1)}$ is the radial wave function of the neutron inside the nucleus).

The energy of the residual nucleus lies in the region of the continuous spectrum. In this case, the stripping reaction cross section, with a proton emitted in an element of solid angle do_p and with energy of the nucleus in the interval dE_f, will be

$$d\sigma = (S/v_d)\rho_f dE_f do_p,$$

where ρ_f is the density of the final states of the nucleus. Noting that $dE_f = -dE_p$ and that

$$S = -32\pi\alpha k_n^2 \sum_l \frac{\Gamma_l^n}{\hbar} \sum_m |I_l^m|^2, ,$$

we finally obtain

$$d\sigma = \frac{32\alpha M k_n^2 R^4}{\pi\hbar^2 k_d (k^2 - k_n^2)^2} \times \sum_l (2l+1)\Gamma_l^n \left| j_l(kR)\frac{dh_l^{(1)}(k_nR)}{dR} - h_l^{(1)}(k_nR)\frac{dj_l(kR)}{dR}\right|^2 \rho_{lf} dE_p do_p. \tag{30}$$

Since there is a large number of levels with different l, angles for which the difference $k^2 - k_n^2$ is quite small play the main role in Eq. (30), in which case

$$\{j_l(kR)\,dh_l^{(1)}(k_nR)/dR - h_l^{(1)}(k_nR)\,dj_l(kR)/dR\} \approx i/k_nR^2.$$

The cross section will now be

$$d\sigma = \frac{M}{\pi^2\hbar^2 k_d}\frac{32\pi\alpha}{(k^2-k_n^2)^2}\sum_l (2l+1)\Gamma_l^n \rho_l dE_p do_p. \tag{31}$$

Availing ourselves of the principle of detailed balance, we can connect the partial width of the disintegration of the residual nucleus, Γ_l^n, to the sticking probability of the neutron to the nucleus, ζ_l, and to the density of the final nuclear states, ρ_l [11]:

$$\Gamma_l^n = \zeta_l/2\pi\rho_l. \tag{32}$$

Substituting (32) into (31) and noting that $k^2 - k_n^2 = 2\{\alpha^2 + (\tfrac{1}{2}\mathbf{k}_d - \mathbf{k}_p)^2\}$, we have

$$d\sigma = \frac{M}{2\pi^3\hbar^2 k_d}\frac{8\pi\alpha}{\{\alpha^2 + (^1/_2\mathbf{k}_d - \mathbf{k}_p)^2\}^2} \times \sum_l (2l+1)\zeta_l dE_p do_p. \tag{33}$$

We see that the deuteron energy is about equally divided between the neutron and proton.

In the case of fast neutrons, they may be considered to be absorbed by the nucleus, provided only that the collision parameter is less than the radius of the nucleus. Inasmuch as we are interested in the stripping process, we need consider only neutrons bound with protons which do not interact with the nucleus. If the average distance between the neutron and proton is R_d, then, obviously, these neutrons will have impact parameters of $l\lambda$ ($\lambda = 2/k_d$) included in the interval between $R - R_d$ and R. Because the effective value of l is considerably greater than unity, the sum in (33) may be replaced by the integral

$$\sum_l (2l+1) = \int_{^1/_2 k_d(R-R_d)}^{^1/_2 k_d R} 2l\,dl = {}^1/_2 k_d^2 R R_d, \quad R_d \ll R. \tag{34}$$

Substituting (34) in (33), we have

$$d\sigma = \frac{2RR_d M k_d \alpha}{\pi^2\hbar^2[\alpha^2 + (^1/_2\mathbf{k}_d - \mathbf{k}_p)^2]^2} dE_p do_p. \tag{35}$$

Integrating (35) over the energies and angles of the outgoing protons, we obtain Serber's formula for the total stripping reaction cross section*

$$\sigma = {}^1/_2\pi R R_d. \tag{36}$$

The distributions of outgoing protons as to angles and energies are also in accord with Serber's "transparent model"[9], viz.,

$$d\sigma(\vartheta_p) = \sigma \sqrt{\frac{\varepsilon}{E_d}} \frac{\vartheta_p d\vartheta_p}{(\varepsilon/E_d + \vartheta_p^2)^{3/2}}, \tag{37}$$

$$d\sigma(E_d) = \frac{\sigma}{\pi} \frac{\sqrt{\varepsilon E_d}\, dE_p}{[(E_p - {}^1/_2 E_d)^2 + \varepsilon E_d]}.$$

In conclusion, I wish to express my deep gratitude to Prof. A. I. Akhiezer for his valuable advice and for reviewing the results of this investigation.

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* The passage in the limit to the Serber theory has been investigated by Bhatia[2], but he obtained too large a value for the total cross section.

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Transfer Processes in Conductors with Regard to Nonlinear Effects

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Relaxation processes are analyzed for a system of lattice electrons. The equations derived in a previous paper[1] to take account of the heating of an electron gas in strong electric fields are made more precise. The form of these equations is given for various conditions and the resulting transfer processes are investigated.

IN a previous work[1] we obtained kinetic equations allowing for the possible heating of an electron gas relative to the lattice. In contrast with the equations of Davydov[2], our equations took the quantum statistics of the electrons into consideration, and could therefore be applied to conductors whose electrons exist in a degenerate state. It is necessary to take the degeneracy of the electron gas into consideration at low temperatures for semimetals and for semiconductors with metallic-type conductivity (excepting the typical metals). The heating of an electron gas for good metals wtih attainable current densities first shows up in effects strongly depending on the temperature of the electrons, such as emission[3], for example, as apparently is confirmed by experiment[4].

For "good" metals at high temperatures, current densities for which it is possible to expect deviations from Ohm's law[5] and nonlinearities in other effects, on account of the heating of the electrons, are not yet attainable. For "poor" metals with a smaller number of charge carriers, however, current densities of the required magnitude are perfectly attainable, and corresponding departures have been observed, for example, in bismuth[6]. It is desirable therefore to investigate all combinations of transfer processes in conductors with regard to the heating of the electrons.

1. RELAXATION PROCESSES IN A METAL

Changes in the electron distribution function f as a result of electron-phonon, electron-electron and electron-impurity collisions, and changes in the phonon distribution function N_ω as a result of phonon-electron, phonon-phonon and phonon-impurity collisions are designated in the following manner:

$$[\partial f/\partial t]_{\text{coll}} = \Sigma^{ef}(f,\ N_\omega) + \Sigma^{ee}(f,\ f) + \Sigma^{en}(f), \tag{1.1}$$

$$[\partial N_\omega/\partial t]_{\text{coll}} = \Sigma^{fe}(N_\omega,\ f) + \Sigma^{ff}(N_\omega,\ N_\omega) + \Sigma^{fp}(N_\omega). \tag{1.2}$$

If we look for solutions of the form $f = f_0 + f'$, $N_\omega = N_\omega^0 + N_\omega'$, where $f_0 = f_0(p)$, $N_\omega^0 = N_\omega^0(k)$ are the symmetric parts of the distribution functions, depending only on the absolute values of the electron and phonon momenta and not on their directions, then the expressions (1.1) and (1.2) take the form

$$[\partial f/\partial t]_{\text{coll}} = \Sigma_{00}^{ef}(f_0,\ N_\omega^0) + \Sigma_{10}^{ef}(f',\ N_\omega^0) + \Sigma_{01}^{ef}(f_0,\ N_\omega') + \Sigma_{00}^{ee}(f_0,\ f_0) + 2\,\Sigma_{10}^{ee}(f',\ f_0) + \Sigma_1^{en}(f'), \tag{1.3}$$

$$[\partial N_\omega/\partial t]_{\text{coll}} = \Sigma_{00}^{fe}(N_\omega^0,\ f_0) + \Sigma_{10}^{fe}(N_\omega',\ f_0) + \Sigma_{01}^{fe}(N_\omega^0,\ f') + \Sigma_{00}^{ff}(N_\omega^0,\ N_\omega^0) + 2\,\Sigma_{01}^{ff}(N_\omega^0,\ N_\omega') + \Sigma_1^{fp}(N_\omega'). \tag{1.4}$$

Here we have neglected the terms $\Sigma_{11}^{ee}(f', f')$, Σ_{11}^{ff}, Σ_{11}^{ef}, Σ_{11}^{fe}, since it is presumed that $f' \ll f_0$, and also the terms $\Sigma_0^{ep}(f_0)$, $\Sigma_0^{fp}(N_\omega^0)$, since when electrons and phonons collide with impurities their energies are not changed.

If the usual assumption is made that the distribution functions f_0 and N_ω^0 are equilibrium distributions, with the same temperature, then operators of the form Σ_{00} reduce to zero.

A change in the directed motion of the electrons at the expense of a different kind of transition is expressed by the operators Σ_{10}^{ef}, Σ_1^{ep}, $2\Sigma_{10}^{ee}$; the influence of the latter on the temperature dependence of the electrical conductivity has been analyzed by Landau and Pomeranchuk[7] and also by Ginzburg and Silin[8]. The operator Σ_{01}^{ef} takes account of the "trapping" of electrons by phonons[9]. We will not consider it here.

As a result of the influence of external fields and gradients, the symmetric part of the distribution functions will depart from equilibrium. The form it will take is determined in part by the external field and gradients, and in part by the operators $\Sigma_{00}^{ef}(f_0, N_\omega^0)$, $\Sigma_{00}^{ee}(f_0, f_0)$. The first term plays the principal role in determining the change in the average energy or the effective temperature of the disordered motion of the electrons, since electron-electron interactions cannot change the average electron energy.

There are substantial changes in the average energy of the random motion of the electrons if the time required for the transfer of the energy obtained by the electron from the field is greater than the time required for the transfer of momentum. In order to ascertain the conditions under which it is possible to observe evidence of this effect in metals, we will estimate the order of magnitude of the different terms in (1.3) and (1.4), which are inversely proportional to the time or to the mean free path for the corresponding type of interaction. In making this estimate we will everywhere use the following values for the physical constants of a metal: number of free electrons $n \sim 3 \times 10^{22}$, limiting energy $\epsilon_0 \sim 5 \times 10^{-12}$ erg ~ 3 ev or a degeneracy temperature $T_0 = \epsilon_0/k \sim 3 \times 10^4$ °K (the corresponding velocity $v \sim 10^8$ cm/sec), a Debye temperature $\Theta_d \sim 10^2$ °K, velocity of sound $v_s \sim 5 \times 10^5$ cm/sec.

The second term in (1.3) is determined by the mean free path over which an electron, interacting with a phonon, loses its momentum. For $T \gg \Theta_d$

$$l_{ef} \sim l_{ef}^0 \Theta_d / T \sim v\Theta_d / \omega_m T \qquad (1.5)$$

$$\sim (hv / k\Theta_d)\, \Theta_d / T \sim 10^{-5}\Theta_d / T \sim 10^{-3} / T.$$

Near 300° K the electrical conductivity corresponding to the value chosen for the constant in l_{ef} is

$$\sigma = e^2 nl / mv \sim 5 \times 10^{16} \text{ cgs} \quad = 5 \times 10^4 \ 1/\Omega \text{ cm}.$$

At low temperatures ($T \ll \Theta_d$)

$$l_{ef} \sim (v / a\omega_m)(\Theta_d / T)^5 \qquad (1.6)$$

$$\sim 10^{-7}(\Theta_d / T)^5 \sim 10^{-7}(10^2 / T)^5.$$

The numerical coefficient a can be determined from the ratio of the ideal electrical conductivities at high ($T' \gg \Theta_d$) and at low ($T \ll \Theta_d$) temperatures,

$$l_{ef} / l'_{ef} = \sigma(T) / \sigma(T') = a^{-1}(\Theta / T)^4 T' / T. \qquad (1.7)$$

The value of a can amount to 500. For definiteness we assume $a \sim 100$.

The mean free path for the scattering of phonons by electrons[9] is, for $T \gg \Theta_d$,

$$l_{fe} \sim v_s \tau_{fe} \sim (v_s / \omega_m)\, \varepsilon_0 / k\Theta_d \sim 10^{-5}, \qquad (1.8)$$

and for $T \ll \Theta_d$,

$$l_{fe} \sim (v_s / \omega_m)\, \varepsilon_0 / kT \sim 10^{-3} / T. \qquad (1.9)$$

At low temperatures this value is less than l_{ff} and essentially determines the temperature part of the thermal resistance of the lattice. In order of magnitude it agrees with the value of l_{fe} obtained from the expression for the thermal conductivity, the experimental value of which, for $T < 50$° K, is[10]

$$\varkappa_p \sim 2 \times 10^{-3}\, T^2 \text{ watt/cm-deg}$$

$$\sim 2 \times 10^4\, T^2 \text{ erg/cm-sec-deg}.$$

Noting that the specific heat $c_p \sim n_a k (T/\Theta)^3$, where the number of atoms is $n_0 \sim n$, we obtain $l_{fe} \sim \varkappa_p / c_p v_s \sim 10^{-2}/T$.

At high temperatures, the phonon distribution function is established principally by way of phonon-phonon interactions. For $T \gg \Theta_d$ the mean free path for phonon collisions is

$$l_{ff} = (v_s / \omega_m)\,\varepsilon_0 / kT \qquad (1.10)$$

$$\sim (h / mv_s^2)(\Theta_d / T) \sim l_{ff}^0\, \Theta_d / T \sim 10^{-5}\Theta_d / T$$

(here noting that $\epsilon_0 \sim p^2/2m \sim h^2\omega_m^2/2mv_s^2 \sim (k\Theta_d)^2/2mv_s^2$).

For this value of the mean free path, the thermal conductivity at 300° K is

$$\varkappa_p \sim {}^1/_3 c_p l_{ff}\, v_s \sim {}^1/_3 n_a k l_{ff}\, v_s$$

$$\sim 10^{-6} \text{ erg/cm-sec-deg},$$

which agrees with the experimental value for the thermal conductivity of insulators.

If one considers that equilibrium in the lattice can be established only by way of transfer processes, then at low temperatures the mean free path l_{ff} increases exponentially, in accordance with the fact that there are only a very small number of the larger phonons taking part in this process. If, however, one evaluates the operator Σ^{ff} without regard for this circumstance, which can only decrease the mean free path, then the estimate gives[9]

$$l_{ff} \sim l_{ff}^0\, (\Theta / T)^5 \sim 10^{-5}(\Theta_d / T)^5. \qquad (1.11)$$

The rate at which equilibrium is established with respect to the energies of the disordered motion among electrons and phonons, which is determined by the operator Σ_{00}^{ef}, is inversely proportional to the mean free path over which an energy kT is transferred. For high temperatures $T \gg \Theta_d$ this path length is

$$l_{ef}^{\varepsilon} \sim (v / \omega_m)\, T / \Theta_d \sim l_{ef}^0 T / \Theta_d \qquad (1.12)$$

$$\sim l_{ef}\, (T / \Theta_d)^2 \sim 10^{-5} T / \Theta_d.$$

Here it is clear that for $T \gg \Theta_d$, l_{ef}^{ϵ} is greater than l_{ef} by a factor $(T/\Theta_d)^2 \sim (kT)^2/mv^2\epsilon_0$.

For $T \ll \Theta_d$:

$$l_{ef}^{\varepsilon} \sim (v / \omega_m)(\Theta_d / T)^3 \qquad (1.13)$$

$$\sim a\,(T / \Theta_d)^2 l_{ef} \sim 10^{-5}(\Theta_d / T)^3.$$

At low temperatures this quantity becomes dependent on the electronic part of the thermal conductivity. The coefficient a is taken as ~100, as before, to agree with the experimental value of the thermal conductivity for these temperatures:

$$\varkappa_{ef} \sim c_e l_{ef}^{\varepsilon} v \sim nk\,\frac{kT}{\varepsilon_0}\, l_{ef}^{\varepsilon} v$$

$$\sim 10^{11} / T^2 \text{ erg/cm-sec-deg}.$$

Here, in particular, it is clear that it is possible to estimate a from measurements of $\varkappa_{ef}/\sigma_{ef}$, since

$$a \sim \varkappa_{ef} e^2 n \varepsilon_0 v_s / \sigma_{ef} c_e (kT)^2 v.$$

It follows from (1.13) that l_{ef}^{ϵ} actually remains greater than l_{ef} down to a rather low temperature, $T > \Theta_d\, a^{-1/2} \sim 10°$ K. At still lower temperatures l_{ef} increases much more rapidly than l_{ef}^{ϵ}.

The rate at which the equilibrium distribution of electrons is established, at some given temperature, as a result of electron-electron interactions is determined by the operator $\Sigma_{00}^{ee} \sim v/l_{ee}$, where l_{ee} is the mean free path over which the electrons redistribute their energy, as well as their momentum. Taking account of the statistics we obtain, in order of magnitude[3],

$$l_{ee} \sim 1 / q_{ee} n\,(kT / \varepsilon_0)^2 \sim 3 \times 10^{-3}(10^2 / T)^2, \qquad (1.14)$$

where a value of 10^{-15} cm^2 is taken for the collision cross section q_{ee}.

For completely free electrons, the term $\Sigma_{10}^{ee}(f_1, f_0)$ has to reduce to zero, since interelectronic interactions cannot change the average momentum of the electrons. However, because of the electronic transfer processes which are possible in lattices, there exists a limiting mean free path for such an interaction. As it turns out, its order of magnitude is the same as that of (1.14)[8].

Finally, the mean free path for scattering by impurities and lattice defects is

$$l_{ep} \sim 1 / \pi^2 q n_p \sim 10^{-4} \qquad (1.15)$$

for $q \sim 10^{-15}$ and an impurity concentration $n_p \sim 10^{18}$, that is, $n_p/n_a \sim 3 \times 10^{-5}$.

For convenience in comparing the role of the processes by which equilibrium is established at various temperatures, we write the approximate values of the mean free paths given by Eqs. (1.5)–(1.15) in tabular form. (The relaxation times corresponding to these are $\tau_e \sim l_e v$, $\tau_f \sim l_f v_s$.)

T	l_{ef}	l_{fe}	l_{ff}	l^{ε}_{ef}	l_{ee}	l_{en}
10^4	10^{-7}	10^{-5}	10^{-7}	10^{-3}	3×10^{-7}	10^{-4}
10^3	10^{-6}	10^{-5}	10^{-6}	10^{-4}	3×10^{-5}	10^{-4}
300	3×10^{-6}	10^{-5}	3×10^{-6}	3×10^{-5}	3×10^{-4}	10^{-4}
100					3×10^{-3}	10^{-4}
30	3×10^{-5}	3×10^{-5}	3×10^{-3}	3×10^{-4}	3×10^{-2}	10^{-4}
10	10^{-2}	10^{-4}	1	10^{-2}	3×10^{-1}	10^{-4}
1	10^3	10^{-3}	10^5	10	3	10^{-4}
0.1	$\gg10^3$	10^{-2}				10^{-4}

From the table it is clear that at $T \sim 10^3$ °K and above, energy equilibrium is achieved at the expense of interelectronic interactions, since in this temperature region $l_{ee} \ll l^{\epsilon}_{ef}$. For lower temperatures ($T < 300^{\circ}$) the electron energy distribution is determined principally by electron-phonon interactions ($l^{\epsilon}_{ef} \ll l_{ee}$) with the exception of only very low temperatures of the order of a few degrees, where l^{ϵ}_{ef} again becomes larger than l_{ee}. However, because the corresponding relaxation times are so large, the process of establishing electronic energy equilibrium will be quite slow if the electrons are not able to exchange energy with the boundaries of the sample or of the individual crystals.

Regarding the establishment of the phonon distribution, it is clear from a comparison of the tabulated values of l_{ff} and l_{fe} that at 300° K and above, phonon equilibrium is established at the expense of phonon-phonon interactions, and for lower temperatures is determined principally by the interaction with electrons.

The heating of an electron gas relative to the lattice is sufficient to bring about changes in the kinetic coefficients before the velocity of the directed motion has become equal to that of the disordered motion, if the frequency of collisions of electrons with phonons with the exchange of energy $\sim kT$ ($\nu^{\epsilon} = 1/\tau^{\epsilon}_{ef} = v/l^{\epsilon}_{ef}$) is less than the frequency of all possible collisions in which momentum is transferred*:

$$\nu = v/L = v(1/l_{ef} + 1/l_{ep} + 1/l_{ee}), \quad (1.16)$$

that is, if the inequality $L \ll l^{\epsilon}_{ef}$ is satisfied. For high temperatures $l^{\epsilon}_{ef} \gg l_{ef}$, for the Debye temperatures $l^{\epsilon}_{ef} \sim l_{ef}$, for temperatures still lower than the Debye temperature, and down to 10-20° K, l^{ϵ}_{ef} can again be considered larger than l_{ef} [on account of the coefficient a in Eq. (1.13)]. For the very lowest temperatures, again $l^{\epsilon}_{ef} \ll l_{ef}$. However, at low temperatures $l_{ef} \gg L \sim l_{ep}$ for the impurity concentrations which we selected for $T \lesssim 30^{\circ}$; consequently, from this point of view the heating of an electron gas is possible both at high and at low temperatures, excluding the region of the Debye temperature.

2. ELEMENTARY CALCULATION OF THE TEMPERATURE OF THE ELECTRONS IN AN ELECTRIC FIELD

It is possible to obtain an estimate of the temperature difference between the electrons and the lattice in the precence of a single external electric field from qualitative considerations. Per unit volume and per unit time, the electrons acquire an energy

$$\sigma E^2 = (e^2 nL/mv)\, E^2 \sim (eEL)^2\, n/\varepsilon_0 \tau \quad (2.1)$$

(where ϵ_0 is the limiting energy of the electrons; σ is the conductivity and E is the electric field). The emission of phonons by electrons is proportional to $N_{\omega} + 1$, the absorption to N_{ω}. Therefore, on the whole the electron gives energy to the lattice in $1/N_{\omega}$ of all the collisions. In unit time it gives up an energy $h\omega/N_{\omega}\tau_1$, where τ_1 is the time between separate discrete collisions, in which a phonon of energy $h\omega$ is emitted or absorbed. For high temperatures, $N_{\omega} \sim kT/h\omega \gg 1$, and the induced phonon emission predominates over spontaneous emission. Therefore, the transfer of energy from electrons to phonons and the mean free path of the electrons regardless of their temperature are completely determined by the distribution of phonons whose equilibrium is established at the expense of phonon-phonon interactions, as we have seen. The momentum of a phonon at high temperatures is comparable with that of the electrons, $h\omega/v_s \sim p$. Therefore, the time τ_1 between separate collisions coincides with τ_{ef}. To summarize, if we designate the average energy lost by an electron in the time t by $\Delta\overline{\epsilon}$, we obtain for the energy transferred by an electron to the lattice per unit time

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* As is clear from the Table, l_{ee} begins to compete with l_{ef}, from Eq. (1.16), and can influence the temperature part of the resistance of metals, only for temperatures $T \lesssim 10^{\circ}$; that is, in the region where impurity scattering predominates.

$$\overline{\Delta\varepsilon} / t \sim h\omega / N_\omega \tau_1 \sim (h\omega)^2 / kT\tau_{ef} \sim \varepsilon_0 m v_s^2 / kT\tau_{ef}\,. \tag{2.2}$$

Moreover, if once considers that, for heating which still does not remove the degeneracy ($k\Theta/\epsilon_0 < 1$), according to the Pauli principle only $nk(\Theta - T)/\epsilon_0$ of the total number n of electrons can transfer this energy, where Θ is the temperature of the electrons and T is that of the lattice, then comparison with (2.1) gives

$$\alpha = (\Theta - T) / T \sim (eEL)^2 l_{ef} / m v_s^2 \varepsilon_0 L. \tag{2.3}$$

For low temperatures ($kT \ll k\Theta_d = h\omega_m$) the picture changes. Whenever electrons and phonons exist in equilibrium, practically the only excitations are phonons with energies $h\omega \lesssim kT$, as can be seen from the expression for the number of phonons

$$N_\omega = [\exp(-h\omega / kT) - 1]^{-1}.$$

Therefore, electrons cannot absorb phonons of large frequency. But electrons at the temperature T also cannot emit phonons of large frequency, since transitions of electrons with the emission of a phonon take place in the zone of diffusion $\sim kT$. This fact also makes it possible to calculate the mean free path l_{ef} for the momentum changes at low temperatures. However, in order for electrons to be heated, it is necessary that phonons with energy $h\omega > kT$ be emitted. The absence of such phonons in the spectrum of the equilibrium distribution at the temperature T and the long relaxation time for the phonon-phonon interactions will lead to a disturbance of the phonon equilibrium distribution. Nevertheless, in problems connected with the rapid transfer of energy from the electrons to the lattice, the energy of the lattice oscillations can be characterized by a temperature T, with suitable reservations. On the other hand, the possibility of investigating marked heating of the electrons relative to the lattice at low temperatures is limited by the fact that the electron mean free path is still not only determined by the lattice temperature T, but also depends on the temperature of the electrons, and, consequently, also on the field. Therefore, it is possible to investigate transition processes by introducing a mean free path for the electrons, but the results are correct only with the assumption that $\Theta / T \lesssim 1$. In this case the electron loses energy $k\Theta \sim kT$ in the time

$$\tau_{ef}^{\varepsilon} \sim a (T / \Theta_d)^2 \tau_{ef} \sim a (kT)^2 \tau_{ef} / m v_s^2 \varepsilon_0.$$

Per unit time, a fraction $nk(\Theta - T)/\epsilon_0$ of the electrons give up an energy

$$n \frac{k(\Theta - T)}{\varepsilon_0} \frac{kT}{\tau_{ef}^{\varepsilon}} \sim \frac{\Theta - T}{T} \frac{n m v_s^2}{a\tau_{ef}}.$$

Comparison with (2.1) leads to an expression differing from (2.2) only in the numerical factor a.

3. THE KINETIC EQUATION FOR DETERMINING THE HEATING OF AN ELECTRON GAS

The heating of electrons in transfer processes can be calculated with the aid of the kinetic equations obtained earlier[1] by transforming the integral operator $\Sigma_{00}^{ef}(f_0, N_\omega^0)$ in (1.3) to a differential one. The system of equations for the symmetric part of the distribution function f_0, with respect to the velocity, and for the nonsymmetric part $f' = (1/v)\,\mathbf{v}\cdot\mathbf{f}_1$ has the form

$$\frac{\partial f_0}{\partial t} + \frac{v}{3} \nabla f_1 + \frac{1}{N} \frac{\partial}{\partial \varepsilon} (NS_1) + \Sigma_{ee}(f_0, f_0) = 0, \tag{3.1}$$

$$\frac{\partial \mathbf{f}_1}{\partial t} + v \nabla f_0 + e\mathbf{E} v \frac{\partial f_0}{\partial \varepsilon} + \frac{e}{mc} [\mathbf{H} \times \mathbf{f}_1] + \frac{v}{L} \mathbf{f}_1 = 0, \tag{3.2}$$

where in energy space the energy flow density is

$$S_1 = {}^1/_3 e\mathbf{E}\cdot\mathbf{f}_1 v - S = {}^1/_3 e\mathbf{E}\cdot\mathbf{f}_1 v - (2 m v_s^2 \varepsilon v / al) [f_0 (1 - f_0) / kT + \partial f_0 / \partial \varepsilon]. \tag{3.3}$$

Here ϵ is the energy, $v = \partial\epsilon/\partial p$ is the velocity, m is the effective mass, e is the charge of the electrons,

v_s is the velocity of sound, T is the temperature of the lattice, $l = l_{ef}$ is the mean free path over which the electron loses momentum through interaction with a phonon, and L is the mean free path with respect to all interactions [see Eq. (1.16)]; E and H are the external electric and magnetic fields, C is the velocity of light, N is twice the density of the electron levels (doubled because of the spin degeneracy; for a free electron gas, $Nd\epsilon = (2\pi\hbar)^{-3}\, 8\pi p^2 dp$), and a is the numerical coefficient discussed above (for $T \gg \Theta_d$, $a \sim 1$). Together with the normalization requirement

$$n = \int f_0 N d\varepsilon \qquad (3.4)$$

the system of equations totally determines the electronic distribution function $f = f_0 + (1/v)\,\mathbf{v}\cdot\mathbf{f}_1$.

Equation (3.1) with S_1 taken from (3.3) differs from the corresponding expression obtained previously[1] by a factor $1/a$ in front of the square brackets, which was not considered in the system of equations used in that paper, and also in the last term, which determines the influence of the electron-electron interaction on the rate of change of the energy flow density in energy space. Furthermore, the third term of (3.1) is written in a form which does not assume any concrete form for the dependence of the density of electronic energy levels N on the energy ϵ, or of the energy ϵ on the momentum p; we can easily convince ourselves of this by modifying the equations in the earlier paper by expanding $\Sigma_{00}^{ef}(f_0, N_\omega^0)$ in $\Delta\epsilon/\epsilon^2$ instead of in $\Delta p/p$*.

The essential assumptions made in obtaining this system of equations were the following: 1) the conductor is isotropic; 2) $f_1 \ll f_0$. This is equivalent[5] to the condition $eEL/\overline{\epsilon} \ll 1$. The formal solution shows that this condition is realized for any amount of heating (or for any field) if only the quantity $\delta = 2mv_s^2 L/kTal \ll 1$ [see (2.36) and the first footnote appearing on the bottom of p. 504 of this article], which is satisfied for high, as well as for sufficiently low temperatures; 3) the phonon distribution is assumed to be an equilibrium distribution, and the electron mean free path does not depend on Θ (or on the field). As we have seen, this assumption is justified for any Θ at high temperatures; at low temperatures it is approximately correct but only in the region $\alpha = (\Theta - T)/T \lesssim 1$.

Integrating (3.1) over phase space, we obtain the equation of conservation of charge:

$$(\partial n / \partial t) + \operatorname{div}(\mathbf{j}/e) = 0, \qquad (3.5)$$

and integrating over $\epsilon N d\epsilon$, the equation of conservation of energy

$$(\partial\varepsilon_e / \partial t) + \operatorname{div}\mathbf{W}_e - \mathbf{E}\cdot\mathbf{j} + Q = 0. \qquad (3.6)$$

Here the electric current

$$\mathbf{j} = \frac{e}{3}\int v\mathbf{f}_1 N d\varepsilon, \qquad (3.7)$$

the electronic heat flow

$$\mathbf{W}_e = \frac{1}{3}\int \varepsilon v \mathbf{f}_1 N d\varepsilon, \qquad (3.8)$$

the energy transferred per unit volume per unit time from the electrons to the lattice

$$Q = \int S N d\varepsilon, \qquad (3.9)$$

and the average energy of the n electrons

$$\varepsilon_e = n\overline{\varepsilon} = \int \varepsilon f_0 N d\varepsilon; \qquad (3.10)$$

S has to be substituted from (1.22),

Equations (3.5) and (3.6) have to be complemented by the equation of conservation of energy for the lattice, which can be obtained by integrating the equation for phonons over phase space,

$$\partial\varepsilon_e / \partial t + \operatorname{div}\mathbf{W}_\mathrm{p} - Q = 0 \qquad (3.11)$$

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* Instead of (6, 8, 9) derived earlier[1] we obtain in this case.

$$(a - b) f_0 = \Sigma_{00}^{ef}(f_0, N_\omega^0) = A_1 f_0 \frac{\partial f_0}{\partial\varepsilon} - A_2 f_0 \frac{\partial^2 f_0}{\partial\varepsilon^2} - (1 - f_0)\frac{1}{N}\frac{\partial}{\partial\varepsilon}(NA_1 f_0) - (1 - f_0)\frac{1}{N}\frac{\partial^2}{\partial\varepsilon^2}(NA_2 f_0) = \frac{1}{N}\frac{\partial}{\partial\varepsilon}(NS),$$

where

$$S = -(1 - f_0)\left[A_1 f_0 + \frac{1}{N}\frac{\partial}{\partial\varepsilon}(NA_2 f_0)\right] - A_2 f_0 \frac{\partial f_0}{\partial\varepsilon}.$$

The equality $S = 0$ for $f_0 = f_{00} = [\exp(\epsilon - \mu)/kT + 1]^{-1}$ gives

$$A_1 = \frac{1}{kT}\left[A_2 - \frac{1}{N}\frac{\partial}{\partial\varepsilon}(NA_2)\right].$$

Substituting A_1 into the expression for S, we obtain

$$S = -A_2\{f_0(1 - f_0)/kT + \partial f_0/\partial\varepsilon\}.$$

As in our earlier Eqs. (14) and (17)[1], A_2 is

$$A_2 = \frac{\overline{(\Delta\varepsilon)^2}}{2\tau_1} = \int \frac{(\Delta\varepsilon^2)}{2} W(\varepsilon, \Delta\varepsilon, \vartheta)\, d\Delta\varepsilon\, d\Omega = \frac{2mv_s^2 v\varepsilon}{al},$$

and from the boundary conditions. This makes it possible to determine the temperature of the lattice uniquely for different processes.

4. SOLUTION OF THE KINETIC EQUATIONS. CONSERVATION AND TRANSFER EQUATIONS

Disregarding the term $\partial \mathbf{f}_1 / \partial t$, it is possible to solve Eq. (3.2) with respect to $\mathbf{f}_1$

$$\mathbf{f}_1 = -eL \frac{\mathbf{G} + (eL/cmv)\mathbf{G}\times\mathbf{H} + (eL/cmv)^2\,\mathbf{H}(\mathbf{G}\cdot\mathbf{H})}{1 + (eL/cmv)^2 H^2}, \tag{4.1}$$

where $\mathbf{G}^{-1} = e^{-1}\nabla - \mathbf{E}\,\partial/\partial\epsilon$ is an operator which acts on the function f_0. The equation obtained by substitution of Eq. (4.1) for $\mathbf{f}_1$ into (3.1) can be solved for f_0 if the first two terms in (3.1) are ignored in comparison with the last in the first approximation. This means that the processes being considered are sufficiently slow and the gradients sufficiently small, that is, the field changes insignificantly along either the electron-electron mean free path, or along the mean free path for the transfer of energy from the electron to the lattice. Concerning the relation between the last two possibilities we can consider two extreme cases: the third term is noticeably smaller than the fourth (strong interelectronic interactions), or the reverse.

In the first case the equation $\Sigma_{00}^{ee}(f_0, f_0) = 0$ gives the Fermi distribution

$$f_0 = (e^{(\varepsilon - \mu)/k\Theta} + 1)^{-1} \tag{4.2}$$

with a chemical potential μ and an undetermined parameter Θ. The connection between these two quantities and the number n of electrons and the temperature T of the lattice is established with the aid of the normalization condition (3.4), but their dependence on the coordinates and on the fields is found in the next approximation by substituting Eq. (4.2) for f_0 into Eq. (3.6).

For the reverse case of interelectronic interactions which are small in comparison with those between electrons and phonons, we substitute Eq. (4.1) for f_1 into Eq. (3.3) for S_1 and obtain

$$S_1 = {}^1\!/_3 e(\mathbf{E}\cdot\mathbf{f}_1)v - S = -(2mv_s^2\varepsilon v/al) \tag{4.3}$$

$$\times\{f_0(1 - f_0)/kT + (\partial f_0/\partial\varepsilon)(1 + \alpha_H)\},$$

where

$$\alpha_H = \alpha\frac{1+\beta^2\gamma^2}{1+\beta^2} = \frac{(eEL)^2}{6mv_s^2\varepsilon}\,\frac{al}{L}\,\frac{1+\beta^2\gamma^2}{1+\beta^2}, \tag{4.4}$$

$$\beta = \frac{eL}{cmv}H, \quad \gamma = \cos(\mathbf{E}\cdot\mathbf{H}).$$

The solution of the equation $S_1 = 0$ has the form

$$f_0 = \left(e^{(z-\zeta)/kT} + 1\right)^{-1}, \quad z = \int^{\varepsilon} d\varepsilon/(1 + \alpha_H), \tag{4.5}$$

where ζ is a constant of integration, whose connection with the number n of electrons and the temperature T of the lattice is established with the aid of the normalization condition (as is easily seen, the possibility of normalization is guaranteed by the requirement that α goes to infinity no faster than ϵ). The dependence of ζ and n on the coordinates and on the fields can be established in the next approximation with the aid of the conservation equation (3.6). Knowledge of the function f_0 makes it possible to evaluate the transfer equations (3.3) and (3.8), which together with the conservation equations (3.6) for electrons and (3.11) for phonons completely determine all galvanomagnetic, thermomagnetic and thermoelectric phenomena.

1. We assume that the distribution function has the Fermi form (4.2), and that it is possible to speak of a temperature for the electrons. It is a particularly simple matter to establish the second assumption in general form for the following situations:

a) There is a single stationary field ($\operatorname{div}\mathbf{W}_e = 0$, $\partial\epsilon_e/\partial t = 0$). Substituting (4.2) into (4.3), we obtain

$$Q - \mathbf{Ej} = -\int NS_1\,d\varepsilon = \frac{\Theta - T}{BT} \tag{4.6}$$

$$-\mathbf{Ej} = \frac{\Theta - T}{BT} - \frac{1}{B(\alpha_H)},$$

$$\frac{1}{B} = -\frac{2mv_s^2}{a}\int\frac{\varepsilon vN}{l}\,\frac{\partial f_0}{\partial\varepsilon}\,d\varepsilon, \tag{4.7}$$

$$\frac{1}{B(\alpha_H)} = -\frac{2mv_s^2}{a}\int\frac{\varepsilon vN}{l}\,\alpha_H\,\frac{\partial f_0}{d\varepsilon}\,d\varepsilon.$$

Here for the temperature of the electrons we obtain

$$(\Theta - T)/T = B/B(\alpha_H). \tag{4.8}$$

In the first approximation, for the case $k\Theta/\mu \ll 1$,

$$\frac{\Theta - T}{T} = \alpha_H^0 = \frac{B_0 j^2}{\sigma_0} \tag{4.9}$$

$$= \frac{e^2 j^2}{6(\sigma_0 / L)^2 m v_s^2 \varepsilon_0} \frac{al}{L} \frac{1 + \beta_0^2 \gamma^2}{1 + \beta_0^2},$$

where the index 0 denotes that the corresponding expression is taken at the limit of the Fermi distribution.

For small magnetic fields or for E perpendicular to H, $\mathbf{j} = \sigma \mathbf{E}$ and the expression for the temperature can be written in the form

$$\frac{\Theta - T}{T} = \alpha_H^0 = \frac{(eEL)^2}{6 m v_s^2 \varepsilon_0} \frac{al}{L} \frac{1 + \beta_0^2 \gamma^2}{1 + \beta_0^2}. \tag{4.10}$$

For $|k\Theta / \epsilon_0| \gg 1$ (classical statistics)

$$(\Theta - T)/T = \overline{\varepsilon v \alpha_H / l} \Big| \overline{\varepsilon v / l} \tag{4.11}$$

(where the line denotes average value).

For small magnetic fields and for l = const, L = const, and also for $\epsilon = p^2/2m$, $N \sim \epsilon^{1/2}$, we obtain from the considerations $\alpha_H \sim 1/\epsilon$ and $\epsilon = 3/2\, k\Theta$

$$\frac{\Theta - T}{T} = \frac{2\bar{v}}{m\overline{v^3}} = \frac{3}{4}\alpha_H \tag{4.12}$$

$$= \frac{(eEl)^2}{12 m v_s^2 k\Theta} \frac{al}{L} \frac{1 + \gamma^2 \beta^2}{1 + \beta^2}.$$

When $l \sim \epsilon^{1/2}$ (α_H = const) the temperature $(\Theta - T)/T = 3\alpha_H/2$.

Solving (4.12) for $k\Theta$, we obtain for the temperature of the electrons an equation which, when written in the form*

$$k\Theta = {}^1/_2 \left(kT + \sqrt{(kT)^2 + 2(eEL)^2 / 3\delta}\right) \tag{4.13}$$

agrees with the old results for classical statistics, without accounting for additional scattering by impurities, however. (In the above equation $\delta = 2 m v_s^2 L / kTal$ is the ratio of the energy given up by the electron to the total energy during the time required for the total loss of momentum; for a plasma $\delta = 2m/M$, where M is the mass of the atom.)

b) The magnetic field is absent. With the aid of Eq. (3.7) for j, we eliminate the field **E** from the Eq. (3.8) for the electronic heat flow in coordinate space W_e, and obtain the equation of conservation of energy in the form

$$\frac{\partial \varepsilon_e}{\partial t} + \frac{\Theta - T}{BT} = \frac{\mathbf{j}^2}{\sigma} - \frac{j}{e}\Theta \nabla \chi + \nabla (\varkappa_e \nabla \Theta), \tag{4.14}$$

where $Q = \int SN d\epsilon = (\Theta - T)/BT$ is the energy given to the phonons by the electrons, σ is the electrical conductivity, χ is the thermal electromotive force, and $\varkappa_e$ is the electronic thermal conductivity *. The coefficients σ, χ, $\varkappa_e$ are expressed in terms of f_0 in the usual way, except with T replaced by Θ. For $\Theta = T$, Eq. (4.14) coincides with the standard equation in the theory of metals. The energy lost from the conductor because of the current passing through it is $\Omega = \int (W_e + W_p) dS$, where the integral is taken over the surface of the conductor.

With the help of Eqs. (4.14) and (3.11), and the transfer equations (3.7) and (3.8), we can derive all the thermoelectric effects related to the heating of the electron gas.

The expression for the temperature of the electrons in the stationary case in the absence of spatial gradients, which is obtained from Eq. (4.14), agrees with Eq. (4.8), in which it is necessary to set the magnetic field equal to zero ($\alpha_H = \alpha$).

2. If the distribution function has the form (4.5), then from the normalization condition (3.4) we can find ζ, which, as simple transformations will show, is equal to the following expression in the second approximation for the case $kT(1 + \alpha)/\epsilon \ll 1$:

$$\zeta = z_0 + \Delta z = z_0 - \frac{(\pi kT)^2}{6}\left\{\frac{1}{N}\frac{d}{d\varepsilon}[N(1 + \alpha)]\right\}_0, \tag{4.15}$$

$$z_0 = \int^{\varepsilon_0} \frac{d\varepsilon}{1 + \alpha_H}.$$

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* Here it is clear that even for strong fields which totally remove the degeneracy and make $\Theta \gg T$, the inequality $eEL/\bar{\epsilon} \sim eEL/k\Theta \sim \sqrt{\delta} \ll 1$ is satisfied if only $\delta \ll 1$.

* The electronic thermal conductivity is determined here, as well as at low temperatures, by the mean free path for momentum transfer (and not by the mean free path for energy transfer, as is usual), since in transforming the term $\Sigma_0^{ef}(f', N_\omega^0)$ in our earlier paper[1] we did not take energy transfer into account, and consequently ignored the temperature part of the thermal resistance at low temperatures, a procedure which is justifiable only in the vicinity of the residual resistance.

We are now in a position to calculate the kinetic coefficients in the transfer equations. The calculations show that, in the absence of spatial gradients, the results obtained in the first approximation in $kT(1+\alpha)/\epsilon$ with the aid of the nonequilibrium distribution function (4.5) agree exactly with the results of the first approximation in $k\Theta/\mu$, obtained with the aid of a distribution function similar to the equilibrium function (4.2) with a temperature Θ, determined from (4.9). Consequently, in this approximation for the galvanomagnetic phenomena, one is unable to say anything about the dependence of α_H [in Eq. (4.5)] on energy. The form of the electron distribution, which depends on the relation between electron-electron and electron-phonon interactions, plays a secondary role in comparison with the primary effect of the heating of the electrons, which is determined by the field and by the electron-phonon interaction. Therefore, even for large heating, say for $kT(1+\alpha)/\epsilon \gg 1$ (obviously, this case cannot be realized for metals), the results obtained with the help of the functions (4.2) and (4.5) differ little.

In the presence of spatial gradients, the difference in the forms of the distribution functions (4.2) and (4.5) can be told in advance from the magnitude of the thermoelectric coefficients. This becomes apparent if one considers that thermoelectric phenomena are rather minute effects in the theory of metals, which appear only in the absence of total degeneracy. Here also the difference did not amount to enough that it was necessary to compute it for a qualitative study of the primary effect of the influence of the heating on the magnitude of the thermoelectric coefficients. For example, the equation for the thermal electromotive force, determined as the coefficient of ∇T [in Eq. (4.14) the coefficient was derived with reference to the electronic temperature Θ] takes the form, for the case (4.2),

$$\chi_T = \chi(1+\alpha^0) = \frac{\pi^2 k^2 T(1+\alpha^0)^2}{\varepsilon}\left\{\frac{\varepsilon}{LvN}\frac{d}{d\varepsilon}(LvN)\right\}_0 \tag{4.16}$$

[$\Theta = T(1+\alpha^0)$ is the temperature of the electrons], and for the case (4.5)

$$\chi'_T = \frac{\pi^2 k^2 T(1+\alpha^0)^2}{\varepsilon_0}\left\{\frac{\varepsilon}{LvN(1+\alpha)}\frac{d}{d\varepsilon}[LvN(1+\alpha)]\right\}_0 \tag{4.17}$$

From the above it is clear that for understanding the principal characteristics of most of the effects, it is quite possible to use a distribution function of the form (4.2), even though the equilibrium distribution may not in fact be realized. At the same time, calculations carried out with the function (4.2) are made considerably easier.

5. GALVANOMAGNETIC AND THERMOELECTRIC PHENOMENA

Before proceeding with an investigation of these phenomena, let us say a few words about the character of the hypotheses which have been made and the experimental conditions which they presuppose.

For the determination of the temperature of the electrons in the case of a single stationary field [Eq. (4.8)] we implicitly assumed that the temperature of the lattice, T, was given, and did not consider Eq. (3.11) at that time, since it follows from this equation that

$$(\Theta - T)/BT = \operatorname{div}\mathbf{W}_p = -\nabla(\varkappa_p\nabla T),$$

that is, the total flow of energy given up by the electrons to the lattice has to be able to be removed by thermal conduction in the lattice. On the other hand, in throwing away the term div $\mathbf{W}_e$ in Eq. (3.6) we actually assumed that the inequality $\nabla(\varkappa_p\nabla T) \gg \nabla(\varkappa_e\nabla\theta)$ was satisfied. With the weakening of this inequality, the relative heating of the electrons decreases, and in the case of the reverse inequality there is no noticeable heating. A similar relation can be satisfied either in the case $\nabla\Theta \to 0$, that is, when the transfer of energy by electrons to an external medium at the boundaries of the conductor is somehow hampered, or in the case $\varkappa_p \gg \varkappa_e$. For good conductors, it is known that the latter inequality is not satisfied, not only for high, but also for low temperatures in the region of residual resistance, because in this region

$$\varkappa_e \sim c_e l_{ep} v \sim nk(kT/\varepsilon_0)\, l_{ep} v$$

$$\gg \varkappa_p \sim c_p l_{fp}\, v_s \sim n_a (T/\Theta_d)^3\, l_{fp}\, v_s.$$

Furthermore, the current density for which the relative heating of electrons $(\Theta - T)/T$ remains appreciable is so large, in the case of metals, that the heat evolved simply cannot be removed from the surface of the conductor. The latter difficulty can to some extent be ameliorated by placing the conductor in liquid helium, which possesses a very large thermal conductivity. However, this does not save the situation, in view of the inequality $\varkappa_p \ll \varkappa_e$ in the region of residual resistance. Consequently, it is impossible to obtain noticeable related heating of electrons in the stationary case for metals. For semiconductors with classical statistics the inequality $\varkappa_e \ll \varkappa_p$ leads to the requirement $n \ll n_a l_{ff} v_s / l_{ef} \sim v_s n_a / v$ and is realized for many semiconductors.

In the nonstationary case the equation for the temperature Θ (4.8) is somewhat changed. The temperatures Θ and T are determined from Eqs. (3.6) or (4.14) and (3.11), which, considering that $\epsilon_e = c_e\Theta$, $\epsilon_p = c_p T$, take the following form in the absence of heat exchange with the surroundings:

$$c_e \partial\Theta / \partial t + (\Theta - T) / BT = j^2 / \sigma, \tag{5.1}$$

$$c_p \partial T / \partial t - (\Theta - T) / BT = 0.$$

For squared current impulses lasting for a length of time $t \gg \tau_T = c_p c_e TB/(c_p + c_e)$ (for metals $\tau_T \sim 10^{-13}$ sec), along with the assumption that c_e, c_p, j^2/σ and BT do not depend on the temperature, the solution of the system of Eq. (5.1) gives

$$T - T_0 = Rj^2 t / (c_p + c_e), \tag{5.2}$$

$$\Theta - T_0 = Rj^2 / (c^p + c_e), \tag{5.3}$$

$$\Theta - T = TBRj^2 c_p / (c_p + c_e), \tag{5.4}$$

where T_0 is the initial temperature, and $R = 1/\sigma$. Equation (5.4) remains valid also if c_p, c_e, etc., do depend on temperature, provided only that $c_p \gg c_e$. This formula differs from Eq. (4.8) only in the factor $c_p/(c_p + c_e)$, which for metals can be markedly less than unity only at low temperatures. Thus for the momentum process the relation $c_p \gg c_e$ is the most advantageous one for realizing electron heating.

We will state some final results for the effects mentioned in the heading of this Sec:

1. *Electrical conductivity.* If we calculate the temperature correction for $k\Theta/\mu \ll 1$ in the usual way, and recall that the results obtained for the galvanomagnetic effects, obtained using Eqs. (4.2) and (4.5), are identical here, we obtain

$$(\sigma - \sigma_0) / \sigma_0 = (k\Theta / \varepsilon_0)^2 \Lambda_{10} \tag{5.5}$$
$$= (kT)^2 (1 + \alpha^0)^2 \Lambda_{10} / \varepsilon_0,$$

where σ_0 is the electrical conductivity for weak currents, α^0 is determined from (4.9) with $H = 0$, and

$$\Lambda_p = \frac{\pi^2}{6} \frac{\varepsilon^2}{LvN\varepsilon^{n-1}} \left\{ \frac{d^2}{d\varepsilon^2} (LvN\varepsilon^{n-1}) - \frac{1}{N} \frac{dN}{d\varepsilon} \frac{d}{d\varepsilon} (LvN\varepsilon^{n-1}) \right\} \tag{5.6}$$

is a coefficient of order of magnitude unity under ordinary conditions.

As already discussed in previous papers[2,5], deviations from Ohm's law, according to Eq. (5.5), can be approached for experimentally attainable current densities only for poor metals like bismuth.

2. *Resistance changes in a magnetic field.* We can obtain a relation similar to (5.5):

$$\frac{\sigma - \sigma_0}{\sigma} = \left[\frac{(kT)^2 (1 + \alpha_H)^2}{\varepsilon^2} \frac{\Lambda_1^1 + \beta^2 (2\Lambda_1^2 - \Lambda_1^1)}{1 + \beta^2} \right]_0 \tag{5.7}$$

where the coefficients Λ_n^m differ from Eq. (5.6) for Λ_n *by* a factor $1/\beta^{m-1}$ outside the curly brackets and a factor $\beta^{n-1}/(1 + \beta^2)$ in the parentheses ($\beta = eLH/cmv$).

For small currents, $\alpha_H = 0$, the result agrees with the well-known result obtained for the isotropic model of a metal in the usual theory, which predicts a change in the electrical conductivity in a magnetic field for $kT\beta/\epsilon \sim 1$. An accurate estimate, which answers to experimental fact, is obtainable only from an anisotropic model (nonspherical Fermi distribution in momentum space) and indicates a magnetc field dependence for $\beta \sim 1$. Our original equations were not applicable to the anisotropic model, and consequently we cannot say whether or not Eq. (5.7) is changed in this case. However, the fact that α_H and β in Eq. (5.7) correlate suggests that in an anisotropic model in a magnetic field, deviations from Ohm's law might set in sooner than the isotropic model would predict; namely, for $\alpha_H \sim 1$ instead of for $\alpha_H kT/\epsilon \sim 1$. Furthermore, for those metals for which the change of electrical conductivity in a magnetic field does not undergo saturation, the factor in (5.7) depending on the magnetic field can become large enough to reduce the magnitude of the current, so that we might expect a dependence of the electrical conductivity on the current.

For strong currents, changes in the Hall constant are determined by a formula similar to (5.7).

For thermoelectric phenomena the temperature of the electrons for weak currents is determined from the equation [see (4.14) and (4.16)]

$$\begin{aligned}\alpha &= (\Theta - T)/T \qquad (5.8)\\ &= Bj^2/\sigma - (Bj\Theta/e)\,db\Theta/dx = \alpha_{\mathrm{tr}} + \alpha_{\mathrm{th}},\end{aligned}$$

where the thermal electromotive force $\chi = b\Theta$ and

$$b = \frac{\pi^2}{3}\left[\frac{k^2}{LvN}\right]\frac{d}{d\varepsilon}(LvN)\Big]_0$$

for the case $k\Theta/\mu \ll 1$.

Since the thermoelectric phenomena appear as second order effects in the theory of metals, vanishing for complete degeneracy, their deviations from linearity have already set in for $\alpha \sim 1$, instead of for $\alpha kT/\epsilon_0 \sim 1$, as in the case of the electrical conductivity.

3. *The Thomson coefficient* [b = const, $\Theta = \Theta(x)$].

$$\zeta = -Q_{\mathrm{T}}/j\frac{d\Theta}{dx} = b\Theta/e \qquad (5.9)$$

will depend on the current for $\alpha \sim 1$. However, for such a value of α the required currents are already so large that in Eq. (5.8) $\alpha_{\mathrm{tr}} \gg \alpha_{\mathrm{T}}$. This circumstance makes it impossible for us to detect the influence of changes in the direction of the current on the temperature.

4. *The Peltier coefficient* [Θ = const, $b = b(x)$].

$$\varPi_{12} = -\frac{1}{j}\int_{(1)}^{(2)} Q_\pi\,dx = \frac{\Theta^2}{e}\Delta b, \qquad (5.10)$$

where $\Delta b = b_2 - b_1$ is the change in the quantity b over the region Δx in going from one metal to the other. The temperature of the electrons in this region is determined from Eq. (5.8), in which it is necessary to set $/db\Theta/dx \sim \Theta\Delta b/\Delta x$. In contrast with the Thomson effect, in the Peltier effect the coefficient α_π can approach unity for currents for which α_{tr} is still small compared with 1. This occurs because of the smallness of the transition region Δx. However, the impossibility of controlling the temperature in the momentum system over such a small region as Δx apparently deprives the Peltier effect of this advantage in practice.

5. *The resistance of a polycrystal.* It appears that there is a medium, nevertheless, in which electron heating at the expense of thermoelectric heat can still be observed. This medium is a semiconductor, between whose individual grains a contact potential difference can exist. By a simple analog, this structure can serve as a closed circuit, consisting of fragments of two dissimilar metals. If the contacts exist at the same temperature, then the contact resistance of such a circuit will be zero, in accordance with the fact that the thermoelectric effects at different junctions have opposite sign. However, precisely because of this difference, the temperature of the electrons in the region of contact will, depending on the sign of α_{th}, either be greater or less than the average temperature of the electrons in the conductor. Determining the potential difference over the region h, inside of which there are two contacts,

$$V_h = \int_h E\,dx = \int_h (j/\sigma + (\chi/e)d\Theta/dx)\,dx$$

[for a closed circuit $\int\Delta\mu\,dx = 0$ and, consequently, $\int(\Delta b)\,dx = 0$] we obtain, for $\alpha_\pi \ll 1 + \alpha_{\mathrm{tr}}$

$$V_h = jhR_0\left[1 + \frac{2B}{h\Delta xR_0}(\varPi^0_{12})^2(1 + \alpha_{\mathrm{tr}})^3\right], \qquad (5.11)$$

$$\varPi^0_{12} = \varPi_{12}(T).$$

Applied to a polycrystal, (5.11) gives

$$\begin{aligned}\frac{\sigma_0 - \sigma}{\sigma} = \frac{R - R_0}{R} = \frac{\Delta R}{R} \qquad (5.12)\\ = \frac{2B(\varPi^0_{12})^2\sigma_0}{\Delta x\,\Delta y}(1 + Bj^2R_0)^3,\end{aligned}$$

where Δy is the linear dimension of an individual grain, Δx is the width of the transition region between them. Actually, on account of the thermal conductivity, which we did not take into consideration, this region will expand, although it will not exceed the dimensions of Δy.

For $\alpha_{\mathrm{tr}} \ll 1$ the additional resistance will not depend on the current density. If we take $\Pi^0_{12} \sim (kT)^2/e\epsilon_0$, then for low temperatures (for $\Theta_{\mathrm{d}} < T < T_0$ the effect is insignificant) we obtain in order of magnitude

$$\frac{\Delta R}{R} \sim \frac{kT}{mv_s^2}\left(\frac{T}{T_0}\right)^3\frac{all_{\mathrm{ep}}}{\Delta x\Delta y} \sim \frac{l_{\mathrm{ep}}vh}{\Delta x\Delta y\,mv_s^2}\left(\frac{\Theta_{\mathrm{d}}}{T_0}\right)^2\frac{\Theta_{\mathrm{d}}}{T} \qquad (5.13)$$

[noting that $l = l_{ef} \sim (vh/ak\Theta_d/T)$]. The quantity $l_{ep}vh/\Delta x\,\Delta y m v_s^2 \sim 10^{-1}$ [for $l_{ep} \sim 10^{-4}$, $v \sim 10^8$, $v_s^2 \sim 10^{11}$, $\Delta x \sim \Delta y \sim 10^{-3}$), so that $\Delta R/R \sim 1$ for $T/\Theta_d \sim (\Theta_d/T_0)^3$.

This region of temperature is attainable only for conductors with low degeneracy temperatures (i.e., with small numbers of free electrons). Thus, for poor conductors, in the momentum regime, one can attempt to observe a reversal of the temperature dependence of the resistance in the low temperature region.

The theory which has been developed is entirely applicable to semiconductors, which in a series of cases have a region of temperatures in which the conduction electrons are degenerate, and for which it is impossible to use classical statistics. Recently, more and more evidence has appeared for the existence of these regions at low temperatures, even for pure semiconductors[11]. For such semiconductors an increase in resistance with decrease in temperature can also arise in the stationary case at low temperatures, in accordance with Eqs. (2.12) and (2.13) ($\varkappa_p \ll \varkappa_e$ at low temperatures because of the small number of electrons), in spite of the fact that the conductor ought to possess metallic-type conductivity in the region of degeneracy.

An investigation of the electrical properties of semiconductors in strong fields in the region of degeneracy has an advantage over an investigation in the region where classical statistics apply, in that in the first case it is much simpler to solve the problem of the number of free carriers of charge, whose increase with field apparently sets in long after the field has removed the degeneracy. This situation contrasts with that for the high temperature region, where classical statistics are obeyed, where it is impossible to be absolutely certain that the increase in the number of free electrons with field shows up in the electrical conductivity and in other effects later than in the relative increase of the temperature of the electrons. On the contrary, there is reason to believe that the hypothesis, on which other work has been based[2,12], concerning the invariability of the number of charge carriers with field down to the critical fields where Ohm's law sets in, is in general wrong[5].

In conclusion, I express my profound gratitude to V. L. Ginzburg for his interest in the work and for his valuable comments.

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Two Electron Charge Exchange of α-Particles in Helium

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The cross section for capture of two electrons by a fast α-particle colliding with a helium atom is calculated in the Born approximation.

THE effect of charge exchange, which occurs when positive ions pass through matter, has been studied thoroughly, both experimentally and theoretically. But in most cases, only collisions with single electron charge exchange have been considered. Recently, Fogel', Krupnik and Safranov[1], in the study of the processes of charge exchange of protons in gases, have observed the capture of two electrons in a single elementary process. They found a value of $\sigma = 2.3 \times 10^{-18}$ cm ^{2} for the cross section of a 21 kev proton transformed into a negative H^- ion in the collision with a hydrogen molecule.

The theory of multiple charge exchange can be treated by the same approximation methods as those used for the theory of atomic collisions;[2] these permitted the computation of single electron processes. For instance,[3] we used the method of perturbing the stationary states to determine the cross section for the simplest two-electron process $H^+ + He \rightarrow H^- + He^{++}$ in the case where the velocity of the proton is much smaller than the velocity of the electrons in the atom. In the present paper, the opposite limiting case of fast collisions is considered, and the Born approximation is applied.

Let us consider the case of the collision of a two-electron atom containing nucleus 1 (atomic number Z_1 mass number A_1) with a nucleus 2 (Z_2, A_2); as a result of the collision, both electrons are captured by 2. If we separate the center-of-mass motion, the Hamiltonian of the problem can be written in the form*

$$\hat{H} = -(1/2\mu_2)\,\Delta_\rho + \hat{H}_2 \qquad (1)$$
$$- Z_1 (1/r + 1/r') + Z_1 Z_2/|\mathbf{r} - \mathbf{s}|;$$

where $\mathbf{r}$ and $\mathbf{r}'$ ($\mathbf{s}$, $\mathbf{s}'$) are the radius vectors of the electrons with respect to the nucleus 1 (2); ρ is the radius vector of the center of mass of the two-electron atom 1 with respect to the nucleus 2; $\mu_2 = MA_1(A_2 + 2\epsilon)(A_1 + A_2 + 2\epsilon)^{-1}$ is the reduced mass ($\epsilon = M^{-1}$ is the ratio of the electron mass and the nucleon mass); and $\hat{H}_2$ is the Hamiltonian of the two-electron atom 2

$$\hat{H}_2 = -\frac{1}{2}\left(1 + \frac{\varepsilon}{A_2}\right)(\Delta_{\mathbf{s}} + \Delta_{\mathbf{s}'}) \qquad (2)$$
$$-\frac{\varepsilon}{A_2}(\nabla_{\mathbf{s}} \nabla_{\mathbf{s}'}) - Z_2\left(\frac{1}{s} + \frac{1}{s'}\right) + \frac{1}{|\mathbf{s} - \mathbf{s}'|}.$$

We denote by $\chi_n^{(2)}(\mathbf{s}, \mathbf{s}')$ the orthonormal eigenfunctions of the operator $\hat{H}_2$

$$\hat{H}_2 \chi_n^{(2)} = W_n^{(2)} \chi_n^{(2)}.$$

The function $\chi_0^{(2)}$ describes the ground state of the atom 2, and $W_0^{(2)}$ is the binding energy of the electrons in this state.

We seek a solution of the Schrödinger equation $\hat{H}\Psi = W\Psi$ in the form

$$\Psi = \sum_n F_n(\rho)\, \chi_n^{(2)}(\mathbf{s}, \mathbf{s}'). \qquad (3)$$

We have the following equation for $F_0(\rho)$

$$(\Delta_\rho + k_2^2) F_0(\rho)$$
$$= -2\mu_2 \iint d\mathbf{s}\, d\mathbf{s}'\, \chi_0^{(2)*}(\mathbf{s}, \mathbf{s}')\, V(\rho, \mathbf{s}, \mathbf{s}')\, \Psi(\rho, \mathbf{s}, \mathbf{s}'),$$

$$V = Z_1 (1/r + 1/r' - Z_2'/|\mathbf{r} - \mathbf{s}|), \qquad (5)$$

$$k_2^2 = 2\mu_2 (W - W_0^{(2)}) = 2\mu_2 E_2; \qquad (6)$$

where E_2 is the kinetic energy of the outgoing particles in the center-of-mass system. We are interested in the solutions of (4) which behave, at large ρ, as $F_0(\rho) \sim \rho^{-1} e^{ik_2\rho} f(\vartheta)$. Obviously,

$$f(\vartheta) = \frac{\mu_2}{2\pi} \iiint d\rho\, d\mathbf{s}\, d\mathbf{s}'\cdot$$
$$\times e^{-i\mathbf{k}_2\rho}\, \chi_0^{(2)*}(\mathbf{s}, \mathbf{s}')\, V(\rho, \mathbf{s}, \mathbf{s}')\, \Psi(\rho, \mathbf{s}, \mathbf{s}') \qquad (7)$$

and the cross section for the capture of the two electrons into the ground state of atom 2 is equal to

$$d\sigma = (k_2/k_1)\, |f(\vartheta)|^2\, d\Omega. \qquad (8)$$

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*All the magnitudes are expressed in the system where $\hbar = e = m = 1$.

The above equations are exact; later we make the following approximations:

1. Born approximation: the wave function in (7) is replaced by the expression

$$\Psi \approx e^{i\mathbf{k}_1\sigma}\chi_0^{(1)}(\mathbf{r}, \mathbf{r}'), \tag{9}$$

where σ is the radius vector of the incoming nucleus 1 with respect to the center of mass of the two-electron atom 2.

2. The following approximate expressions are used for the ground state wave functions of the atoms 1 and 2

$$\chi_0^{(1)} = (\alpha_1^3/\pi)\, e^{-\alpha_1(r+r')}, \qquad \chi_0^{(2)} = (\alpha_2^3/\pi)\, e^{-\alpha_2(s+s')}. \tag{10}$$

The criterion for the validity of the Born approximation is usually given by the inequality $Z_1Z_2/v << 1$ but the study of the single electron process $H^+ + H \rightarrow H + H^+$ has shown[4] that the Born approximation gives results in good agreement with the experimental data for smaller velocities, down to $v \sim 1$. Apparently this is related to the fact that in the initial state as well as in the final one, one of the colliding particles is neutral, and the Coulomb interaction practically vanishes for distances somewhat larger than the atomic radius. Therefore a plane wave is a sufficiently good description of the motion of the atom and of the ion, even for not very high energies.

The same situation takes place in the case of a two-electron capture by the process

$$\mathrm{He}^{++} + \mathrm{He} \rightarrow \mathrm{He} + \mathrm{He}^{++}. \tag{11}$$

In this case, the criterion for the validity of the Born approximation is not obvious. If one requires $4v \leq 1$, then $E_{\mathrm{lab}} \gtrsim 1.5$ mev; but if one relaxes the condition to $v_0 \lesssim v$ (where $v_0 = \alpha$ is the mean velocity of the electrons in the helium atom), the condition on the energy becomes weaker: $E_{\mathrm{lab}} \gtrsim 0.3$ mev. (Note that in the case of proton charge exchange in hydrogen, both criteria coincide: $v \sim 1$.) Notwithstanding the fact that the nuclei 1 and 2 are identical in the process (11), the symmetrization of the wave function is not necessary because, for high energies, the nuclei are practically distinguishable.

For the sake of simplicity, all the following formulas refer to the process (11); their generalization to the case of arbitrary Z and A does not present any difficulty.

After an obvious change of integration variables, the amplitude $f(\vartheta)$ for a double charge exchange of α-particles in helium is equal to

$$f(\vartheta) \approx 4(J_1 - J_2)/\pi^3\varepsilon\alpha^2; \tag{12}$$

$$J_1 = \int d\mathbf{R}\, e^{i\mathbf{gR}} \int \frac{d\mathbf{r}}{r} e^{-i\varkappa\mathbf{r} - r - |\mathbf{r}-\mathbf{R}|} \times \int d\mathbf{r}'\, e^{-i\varkappa\mathbf{r}' - r' - |\mathbf{r}'-\mathbf{R}|}, \tag{13}$$

$$J_2 = \int \frac{d\mathbf{R}}{R} e^{i\mathbf{gR}} \left(\int d\mathbf{r}\, e^{-i\varkappa\mathbf{r} - r - |\mathbf{r}-\mathbf{R}|}\right)^2; \tag{14}$$

$$\mathbf{g} = \frac{1}{\alpha}\left(\mathbf{k}_1 - \frac{\mathbf{k}_2}{1+\varepsilon/2}\right), \quad \varkappa = \frac{\varepsilon(\mathbf{k}_1 + \mathbf{k}_2)}{4\alpha(1+\varepsilon/2)}, \tag{15}$$

$$k_1^2 = k_2^2 = k^2 \approx (2/\varepsilon) E_{\mathrm{lab}}, \quad \times \cos\vartheta = (\mathbf{k}_1\mathbf{k}_2)/k^2, \quad \alpha = 1.69. \tag{16}$$

Let us express the integrals J_1 and J_2 in a form suitable for numerical integration. We represent

$$Q(\mathbf{R}) = \int d\mathbf{r} \exp(-i\varkappa\mathbf{r} - r - |\mathbf{r} - \mathbf{R}|)$$

in a Fourier integral

$$Q(\mathbf{R}) = \frac{8}{\pi}\int d\xi\, e^{i\xi\mathbf{R}} (1+\xi^2)^{-2}[1 + (\xi + \varkappa)^2]^{-2},$$

using the identity[5]

$$\frac{1}{a^2b^2} = \int_0^1 \frac{6x(1-x)\,dx}{[ax + b(1-x)]^4},$$

letting $a = 1 + (\xi + \varkappa)^2$, $b = 1 + \xi^2$; we denote

$$p^2 = \varkappa^2 x(1-x) + 1, \quad q^2 = \varkappa^2 y(1-y) + 1. \tag{17}$$

We then have

$$Q(\mathbf{R}) = (48/\pi)\int_0^1 dx\,(1-x)\,x e^{-ix\varkappa\mathbf{R}} \times \int (\eta^2 + p^2)^{-4}\, d\eta\, e^{i\eta\mathbf{R}}$$

$$= 2\sqrt{2\pi} R^{5/2} \int_0^1 dx\,(1-x)\,x p^{-5/2} e^{-ix\varkappa\mathbf{R}} K_{5/2}(pR);$$

the integral J_2 is equal to

$$J_2 = 48\pi^3 \int_0^1\int_0^1 dx\,dy\,xy(1-x) \times (1-y)\,p^{-5}q^{-5}\lambda F_2(x, y), \tag{18}$$

$$F_2 = 3 + 2(2p^2 + 3pq + 2q^2)\lambda \tag{19}$$
$$+ 8(p^4 + 2p^3q + 3p^2q^2 + 2pq^3 + q^4)\lambda^2$$
$$+ 48pq(p^2 + q^2)(p+q)^2\lambda^3 + 128p^2q^2(p+q)^4\lambda^4,$$

$$1/\lambda = (p+q)^2 + [g - (x+y)\varkappa]^2. \tag{20}$$

Similar manipulations yield

$$J_1 = 384\pi^3$$
$$\times \int_0^1\int_0^1 dx\,dy\,xy(1-x)p^{-5}q^{-3}\lambda^3F_1(x,y), \tag{21}$$

$$F_1 = q^3 + 2p(p+q)^2(p^2 - 2pq + 3q^2)\lambda \tag{22}$$
$$+ 16p^2q(p+q)^4\lambda^2.$$

It is easy to see that for energies $E_{\text{lab}} \sim 1$ mev, when the Born approximation begins to be valid, each of the magnitudes p, q and $\varkappa^2$ does not exceed a few units, and $k^2 \sim 10^8$; therefore, for $\vartheta >> 1/k \sim 10^{-4}$:

$$\lambda^{-1} = 2(1 + pq) + (1 - x - y + 2xy)\varkappa^2$$
$$+ (4k^2/\alpha^2)\sin^2(\vartheta/2)$$
$$\approx (2k/\alpha)^2\sin^2(\vartheta/2) \gg 1,$$

and $F_2 \approx 3$. Formula (18) becomes

$$J_2 \approx \frac{36\pi^3\alpha^2}{\sin^2(\vartheta/2)}\left(\int_0^1 \frac{dx\,(1-x)\,x}{[\varkappa^2x(1-x)+1]^{3/2}}\right)^2,$$

and since, for $\lambda << 1$ $J_1 << J_2$, the amplitude $f(\vartheta)$ is equal to

$$f(\vartheta) \approx -\frac{4}{\varepsilon k^2\sin^2(\vartheta/2)}\left[1 + \left(\frac{\varepsilon k}{4\alpha}\right)^2\cos^2\frac{\vartheta}{2}\right]^{-4}. \tag{23}$$

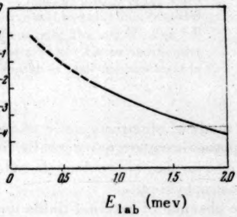
The contribution to the charge exchange cross section σ from angles $\vartheta > \vartheta_0 \sim 10^{-3}$ to 10^{-2} amounts to

$$\frac{64\pi}{\varepsilon^2k^4}\int_0^{1-(\vartheta_0/2)^2} \frac{dx}{(1-x)^2\left[1 + \left(\frac{\varepsilon k}{4\alpha}\right)^2 x\right]^8} \tag{24}$$

$$\approx \frac{256\pi}{\varepsilon^2k^4\vartheta_0^2}\left[1 + \left(\frac{\varepsilon k}{4\alpha}\right)^2\right]^{-8}.$$

For angles $\vartheta < \vartheta_0$ the integrals J_1 and J_2 were obtained by numerical integration.

The results of our calculations of the cross section for the process (11), as a function of the energy of the incoming α-particle in the Laboratory system, are plotted in the graph. When comparing with experiment, one should remember that the capture into excited states of the helium atom has not been taken into account.



We are indebted to Ia. M. Fogel' who brought our attention to the experimental data on double charge exchange.

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On the Oscillations of an Electron Plasma in a Magnetic Field

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The kinetic theory of the oscillations of an electron plasma in a constant magnetic field is examined. An investigation is made of plasma oscillations of frequencies which are integral multiples of the gyromagnetic frequency. The indices of refraction are determined for the ordinary, the extraordinary and the plasma waves which are propagated at an arbitrary angle θ with respect to the magnetic field. It is shown that at frequencies which are integral multiples of the gyromagnetic frequency the plasma wave is highly damped if $\theta < \pi/2$. If $\theta \approx \pi/2$ then the plasma waves corresponding to these frequencies cannot be propagated at all. In this paper the width of the "gaps" in the frequency spectrum of the plasma oscillations is determined.

THE study of electromagnetic processes in electron plasma in an external magnetic field is of interest for a number of problems in radiophysics and in astrophysics.

In the absence of external fields the oscillatory properties of the plasma have been studied in the papers of Vlasov[1] and Landau[2]. A characteristic feature which distinguishes plasma from other media from the point of view of the propagation of electromagnetic waves is the possibility of the existence in the plasma of weakly damped longitudinal electromagnetic waves (plasma oscillations).

The presence of a magnetic field leads to an anisotropy of the properties of the plasma and also gives rise to a number of characteristic resonance effects. The properties of an electron plasma situated in a magnetic field have been investigated on the basis of the kinetic theory first of all by Akhiezer and Pargamanik[3], and later by Gross[4] who showed the existence of bands of forbidden frequencies, which are integral multiples of the gyromagnetic frequency, when the plasma wave is propagated in a direction perpendicular to the direction of the magnetic field. Subsequently, the properties of plasma in a magnetic field have been studies in Refs. 5-7. In particular, Gershman[6] investigated the influence of thermal motion on the propagation of electromagnetic waves in the plasma. The theory of the propagation of electromagnetic waves in plasma is given in the hydrodynamic approximation in the monograph by Al'pert, Ginzburg and Feinberg[8].

The present article is devoted to the investigation of the oscillations of an electron plasma in a magnetic field on the basis of kinetic theory.

1. THE DISPERSION EQUATION

We shall examine the free oscillations of a plasma in a constant and uniform magnetic field **H**. Small oscillations of such a plasma are described by the linearized kinetic equation

$$\frac{\partial f}{\partial t} + \mathbf{v}\frac{\partial f}{\partial \mathbf{r}} + \frac{e}{m}\mathbf{E}\frac{\partial f_0}{\partial \mathbf{v}} + \frac{e}{mc}[\mathbf{vH}]\frac{\partial f}{\partial \mathbf{v}} = 0, \quad (1)$$

where $f(\mathbf{r}, \mathbf{v}, t)$ is a small deviation of the electron distribution function from the Maxwellian function

$$f_0 = n_0 (m/2\pi T)^{3/2} e^{-mv^2/2T},$$

n_0 is the equilibrium electron density, e and m are the charge and mass of the electron, T is the plasma temperature, **E** is the self-consistent electric field, determined by the equation

$$\Delta \mathbf{E} - \operatorname{grad}\operatorname{div}\mathbf{E} - c^{-2}\partial^2\mathbf{E}/\partial t^2 = 4\pi c^{-2}\partial \mathbf{j}/\partial t, \quad (2)$$

where **j** is the electron current density

$$\mathbf{j} = e\int \mathbf{v} f d\mathbf{v}. \quad (3)$$

The properties of electromagnetic oscillations propagated in an unbounded electron plasma after a sufficiently long interval of time after the introduction of the initial disturbance are described by the dispersion equation which relates the frequency of the oscillations ω to the propagation vector **k**. In order to find the dispersion equation we shall look for the solutions of Eqs. (1) and (2) of the form

$$f(\mathbf{r}, \mathbf{v}, t) = f(\mathbf{v}, \mathbf{k}, \omega)\, e^{i(\mathbf{kr}-\omega t)};$$

$$E(\mathbf{r}, t) = E(\mathbf{k}, \omega)\, e^{i(\mathbf{kr}-\omega t)}. \quad (4)$$

Substituting (4) into (1) and (2) we shall obtain a system of equations for the amplitudes $f(\mathbf{v}, \mathbf{k}, \omega)$ and $E(\mathbf{k}, \omega)$:

$$i(\mathbf{kv}-\omega)f + e(\mathbf{Ev})f'_0 - \omega_H \partial f/\partial\vartheta = 0, \quad (5)$$

$$-k^2\mathbf{E} + \mathbf{k}(\mathbf{kE}) + (\omega^2/c^2)\mathbf{E} = -i\frac{4\pi}{c^2}\omega\int \mathbf{v} f d\mathbf{v}. \quad (6)$$

Here $f'_0 = \partial f_0/\partial\epsilon$, $\epsilon = mv^2/2$, $\omega_H = eH/mc$ is the gyromagnetic frequency of the electrons and ϑ is the polar angle in the velocity space (the Z axis is directed along the magnetic field $\mathbf{H}$, and the angle ϑ is measured from the plane containing the vectors $\mathbf{H}$ and $\mathbf{k}$).

The integration of Eq. (5) yields

$$f(\mathbf{v}) = \frac{e}{\omega_H} f'_0 \mathbf{E} \exp\left\{\frac{i}{\omega_H}\int_0^{\vartheta}(\mathbf{kv}-\omega)\,d\vartheta\right\}$$

$$\times\left\{\int_0^{\vartheta}\mathbf{v}\exp\left\{-\frac{i}{\omega_H}\int_0^{\psi}(\mathbf{kv}-\omega)d\psi\right\}d\psi + \mathbf{C}\right\}, \quad (7)$$

where the constant of integration $\mathbf{C}$ is determined from the condition of periodicity $f(\vartheta + 2\pi) = f(\vartheta)$:

$$\mathbf{C} = \int_0^{2\pi}\mathbf{v}\exp\left\{\frac{i}{\omega_H}\int_0^{\psi}(\mathbf{kv}-\omega)\,d\psi\right\}$$

$$\times\, d\psi\Big/\left[1 - \exp\left\{\frac{i}{\omega_H}\int_0^{2\pi}(\mathbf{kv}-\omega)d\vartheta\right\}\right].$$

Substituting (7) into (6) we shall obtain a system of equations which determines the electric field of the plasma waves

$$\sum_{k=1}^{3}\{n^2(\varkappa_i\varkappa_k - \delta_{ik}) \quad (8)$$

$$+ \varepsilon_{ik}\}E_k = 0 \qquad (i = 1,2,3).$$

Here $n = kc/\omega$ is the index of refraction for the wave of frequency ω, $\varkappa_i = k_i/k$ and ϵ_{ik} is the dielectric permittivity tensor of the plasma in the magnetic field which is determined by the expression (see Ref. 7),

$$\varepsilon_{ik}(\omega, \mathbf{k}) = \delta_{ik} + i\frac{4\pi e^2}{\omega\omega_H}\int v_i f'_0 \exp(i\alpha\sin\vartheta + i\beta\vartheta) \quad (9)$$

$$\times\left\{\int_0^{\vartheta} v_k \exp(-i\alpha\sin\psi - i\beta\psi)\,d\psi + C_k\right\}d\mathbf{v},$$

where

$$\alpha = k_x v_r/\omega_H, \quad \beta = (k_z v_z - \omega)/\omega_H.$$

The dielectric permittivity tensor introduced above depends not only on the frequency ω, but also on the propagation vector $\mathbf{k}$, i.e., the plasma is a medium in which the dispersion depends both on space and time[9]. In such media the value of the vector of the electric displacement $\mathbf{D}(\mathbf{r}, t)$ at the point $\mathbf{r}$ and at the instant of time t is determined by the values of the field $\mathbf{E}(\mathbf{r}', t')$ over all space and at all instants of time.

Taking into account the fact that

$$e^{-i\alpha\sin\psi} = \sum_{n=-\infty}^{\infty} J_n(\alpha)e^{-in\psi};$$

$$\int_0^{2\pi} e^{i\alpha\sin\psi - in\psi}d\psi = 2\pi J_n(\alpha), \quad (10)$$

and also using the well known recurrence relations for the Bessel functions $J_n(\alpha)$, we shall write the components of the tensor ϵ_{ik} in the following form:

$$\varepsilon_{11} = 1 - \frac{\Omega^2}{\omega^2}\frac{4z_0}{\sqrt{\pi}} \quad (11)$$

$$\times\sum_{n=-\infty}^{\infty}\frac{n^2}{\lambda^2}\int_0^{\infty} te^{-t^2}J_n^2(\lambda t)\,dt)\int_C \frac{e^{-y^2}}{z_n - y}dy;$$

$$\varepsilon_{12} = -i\frac{\Omega^2}{\omega^2}\frac{4z_0}{\sqrt{\pi}}$$

$$\times\sum_{n=-\infty}^{\infty}\frac{n}{\lambda}\int_0^{\infty} t^2e^{-t^2}J_n(\lambda t)\,J'_n(\lambda t)\,dt\int_c \frac{e^{-y^2}}{z_n - y}dy;$$

$$\varepsilon_{13} = -\frac{\Omega^2}{\omega^2}\frac{4z_0}{\sqrt{\pi}}$$

$$\times\sum_{n=-\infty}^{\infty}\frac{n}{\lambda}\int_0^{\infty} te^{-t^2}J_n^2(\lambda t)\,dt\int_c \frac{ye^{-y^2}}{z_n - y}dy;$$

$$\varepsilon_{22} = 1 - \frac{\Omega^2}{\omega^2}\frac{4z_0}{\sqrt{\pi}}\sum_{n=-\infty}^{\infty}\int_0^{\infty} t^3e^{-t^2}J_n'^2(\lambda t)\,dt\int_c \frac{e^{-y^2}}{z_n - y}dy;$$

$$\varepsilon_{23} = i\frac{\Omega^2}{\omega^2}\frac{4z_0}{\sqrt{\pi}}$$

$$\times\sum_{n=-\infty}^{\infty}\int_0^{\infty} t^2e^{-t^2}J_n(\lambda t)\,J'_n(\lambda t)\,dt\int_c \frac{ye^{-y^2}}{z_n - y}dy;$$

$$\varepsilon_{33} = 1 - \frac{\Omega^2}{\omega^2}\frac{4z_0}{\sqrt{\pi}}\sum_{n=-\infty}^{\infty}\int_0^{\infty} te^{-t^2}J_n^2(\lambda t)\,dt\int_c \frac{y^2e^{-y^2}}{z_n - y}dy;$$

$$\varepsilon_{21} = -\varepsilon_{12}; \quad \varepsilon_{31} = \varepsilon_{13}; \quad \varepsilon_{32} = -\varepsilon_{23},$$

where

$$\lambda = \sqrt{\frac{2}{3}}\,\frac{k_x s}{\omega_H},$$

$$z_n = \sqrt{\frac{3}{2}}\,\frac{(\omega - n\omega_H)}{k_z s}, \quad \Omega = \sqrt{\frac{4\pi n_0 e^2}{m}}$$

$$s = \sqrt{3T/m}, \quad a = \sqrt{T/4\pi n_0 e^2}$$

(a is the Debye radius, Ω is the Langmuir frequency). Integration over y in (11) is carried out along the contour C along the real axis going around the singular point $y = z_n$ on the lower side[2].

The system of equations (8) has a solution different from zero provided its determinant is equal to zero. This dispersion equation, which connects the frequency ω and the propagation vector **k** of the electromagnetic waves in the plasma, can be written in the form

$$An^4 + Bn^2 + C = 0, \tag{12}$$

$$A = \varepsilon_{11}\sin^2\theta + \varepsilon_{33}\cos^2\theta + 2\varepsilon_{13}\cos\theta\sin\theta, \tag{13}$$

$$B = 2(\varepsilon_{12}\varepsilon_{23} - \varepsilon_{22}\varepsilon_{13})\cos\theta\sin\theta$$

$$-(\varepsilon_{22}\varepsilon_{33} + \varepsilon_{23}^2)\cos^2\theta + \varepsilon_{13}^2 - \varepsilon_{11}\varepsilon_{22}$$

$$-(\varepsilon_{11}\varepsilon_{22} + \varepsilon_{12}^2)\sin^2\theta, \quad C = |\varepsilon_{ik}|.$$

θ is the angle between the directions of **H** and **k** ($\varkappa_1 = \sin\theta$, $\varkappa_2 = 0$, $\varkappa_3 = \cos\theta$).

In the general case, the dispersion equation (12) is quite complicated, and therefore we shall restrict ourselves to the examination of the limiting cases of a weak magnetic field ($\omega_H \ll \Omega$) and "low" temperatures ($\omega_H \gg ks$).

2. WEAK MAGNETIC FIELD

In the case of a weak magnetic field ($\omega_H \ll \Omega$) it is convenient for the calculation of ϵ_{ik} to start not with Eq. (11), but directly with Eq. (9). Integrating over ψ in (9) by parts and noting that for $|z| = \sqrt{(3/2)}\,|(\omega/ks)| \gg 1$ and $|\mathrm{Im}\, z| \ll 1$

$$\frac{1}{\sqrt{\pi}}\int\limits_C \frac{e^{-y^2}}{z-y}\,dy \approx \frac{1}{z}\left(1 + \frac{1}{2z^2} + \frac{3}{4z^4} + \dots\right), \tag{14}$$

we obtain for the components of the dielectric permittivity tensor the following expressions

$$\varepsilon_{11} = \varepsilon_0 - k^2a^2v^2(\cos^2\theta + 3\sin^2\theta) - uv; \tag{15}$$

$$\varepsilon_{12} = -\varepsilon_{21} = -i\sqrt{u}\,v;$$

$$\varepsilon_{13} = \varepsilon_{31} = -2k^2a^2\cos\theta\sin\theta v^2;$$

$$\varepsilon_{22} = \varepsilon_0 - k^2a^2v^2 - uv;$$

$$\varepsilon_{33} = \varepsilon_0 - k^2a^2v^2(3\cos^2\theta + \sin^2\theta);$$

$$\varepsilon_{32} = -\varepsilon_{23} \approx 0.$$

Here $\epsilon_0 = 1 - \Omega^2/\omega^2$ is the dielectric permittivity of the plasma in the absence of a magnetic field, $v = \Omega^2/\omega^2$, $u = \omega_H^2/\omega^2$. Using (15) we write the dispersion equation (12) in the form

$$\frac{s^2}{c^2}vn^6 - \left(\varepsilon_0 - uv\sin^2\theta + \frac{8}{3}\frac{s^2}{c^2}v\varepsilon_0\right)n^4 \tag{16}$$

$$+\left[2\varepsilon_0^2 - uv(2\varepsilon_0 + \sin^2\theta) + \frac{5}{3}\frac{s^2}{c^2}v\varepsilon_0^2\right]n^2$$

$$-\varepsilon_0^3 + \varepsilon_0 uv(1 + \varepsilon_0) = 0.$$

Neglecting terms in (16) which are proportional to $s^2/c^2 \ll 1$, we find the indices of refraction for the ordinary and the extraordinary waves

$$n_{1,2}^2 = \frac{2\varepsilon_0 - uv(2\varepsilon_0 + \sin^2\theta) \pm \{[2\varepsilon_0 - uv(2\varepsilon_0 + \sin^2\theta)]^2 - 4(\varepsilon_0 - uv\sin^2\theta)[\varepsilon_0^3 - \varepsilon_0 uv(1+\varepsilon_0)]\}^{1/2}}{2(\varepsilon_0 - uv\sin^2\theta)}. \tag{17}$$

For the index of refraction of the plasma wave n_3 we obtain the following expression in the case $\epsilon_0 \ll 1$:

$$n_3^2 = (\varepsilon_0 - u\sin^2\theta)\,c^2/s^2. \tag{18}$$

If $\epsilon_0 \approx 1$, then the third solution of (16) does not satisfy the condition $|z| \gg 1$, under which Eq. (16) itself has been obtained.

Expressions (17) and (18) hold if the terms in (16) neglected in obtaining (17) and (18) are small compared to the terms which have been retained. For this to hold it is necessary that the following inequalities should be satisfied

$$|\varepsilon_0 - u\sin^2\theta| \gg (s/c)\varepsilon_0,$$

$$(s/c)\sqrt{u\varepsilon_0}, \quad (s/c)\sqrt{u}\,|\sin\theta|.$$

In the opposite case, when

$$(\varepsilon_0 - u \sin^2\theta) \ll (s/c)\sqrt{u}\,|\sin\theta|$$

we have

$$\begin{aligned} n_1^2 &= u, \\ n_2^2 &= \sqrt{u}\,|\sin\theta|\,c/s \quad (|\sin\theta| \gg s/c\sqrt{u}). \end{aligned} \tag{19}$$

In the case $\theta = 0$ and $\epsilon_0 = 0$ all three solutions of (16) reduce to zero.

3. STRONG MAGNETIC FIELD (LOW TEMPERATURES)

We shall now investigate the dispersion equation in the case of low temperatures ($\lambda \ll 1$). Far from the resonance frequencies ($|Z_n| \gg 1$) we may use the asymptotic expression (14) for the integrals along the contour C in (11). Expanding the functions $J_n(\lambda t)$ and $J_n'(\lambda t)$ in series in powers of λ, we obtain the following expressions for the components ϵ_{ik}:

$$\begin{aligned}
\varepsilon_{11} &= 1 - v/(1-u) - k^2a^2v^2\left[\frac{1+3u}{(1-u)^3}\cos^2\theta + 3(1-u)^{-1}(1-4u)^{-1}\sin^2\theta\right]; \\
\varepsilon_{12} &= -iv\sqrt{u}/(1-u) - ik^2a^2v^2\sqrt{u}\,[(3+u)(1-u)^{-3}\cos^2\theta \\
&\quad + 6(1-u)^{-1}(1-4u)^{-1}\sin^2\theta]; \\
\varepsilon_{13} &= -2k^2a^2v^2(1-u)^{-2}\sin\theta\cos\theta; \\
\varepsilon_{22} &= 1 - v/(1-u) - k^2a^2v^2\,[(1+3u)(1-u)^{-3}\cos^2\theta \\
&\quad + (1+8u)(1-u)^{-1}(1-4u)^{-1}\sin^2\theta]; \\
\varepsilon_{23} &= ik^2a^2v^2\sqrt{u}\,(3-u)(1-u)^{-2}\sin\theta\cos\theta; \\
\varepsilon_{33} &= 1 - v - k^2a^2v^2\,[3\cos^2\theta + (1-u)^{-1}\sin^2\theta),
\end{aligned} \tag{20}$$

The terms in (20) which are proportional to $(ka)^2$ take into account the thermal motion of the electrons in the plasma which determines the spatial dispersion of the medium.

Making use of (20) we evaluate the coefficients A, B, C of Eq. (12)

$$\begin{aligned} A &= A_0 + A_1n^2; \\ B &= B_0 + B_1n^2; \quad C = C_0 + C_1n^2, \end{aligned} \tag{21}$$

$$\begin{aligned}
A_0 &= \frac{1 - u - v + uv\cos^2\theta}{1-u}, \\
B_0 &= \frac{(2-v)\,u - 2(1-v)^2 - vu\cos^2\theta}{1-u}, \\
C_0 &= \frac{(1-v)\,[(1-v)^2 - u]}{1-u};
\end{aligned} \tag{22}$$

$$\begin{aligned}
A_1 &= -\frac{s^2}{3c^2}v\left\{3\cos^4\theta + \frac{6-3u+u^2}{(1-u)^3}\cos^2\theta\sin^2\theta + \frac{3}{(1-u)(1-4u)}\sin^4\theta\right\}; \\
B_1 &= \frac{s^2}{3c^2}v\,\Bigl\{\frac{2(1+u-v)}{(1-u)^2}\cos^2\theta\sin^2\theta \\
&\quad + \frac{1+\cos^2\theta}{1-u}\left[(1-u-v)\left(3\cos^2\theta + \frac{\sin^2\theta}{1-u}\right) + (1-v)\left(\frac{1+3u}{(1-u)^2}\cos^2\theta + \frac{3\sin^2\theta}{1-4u}\right)\right] \\
&\quad + \frac{2\sin^2\theta}{(1-u)^2}\left[\frac{1+3u-v-uv}{1-u}\cos^2\theta + \sin^2\theta\,\frac{2(1-u)(1+2u-v)}{1-4u}\right]\Bigr\}; \\
C_1 &= -\frac{s^2}{3c^2}v\,\Bigl\{\frac{2(1--v)}{(1-u)^2}\left[\frac{1+3u-v-uv}{1-u}\cos^2\theta + \frac{(1-u)(1+2u-v)}{1-4u}2\sin^2\theta\right] \\
&\quad + \frac{(1-v)^2-u}{1-u}\left(3\cos^2\theta + \frac{\sin^2\theta}{1-u}\right)\Bigr\}.
\end{aligned} \tag{23}$$

The coefficients A_1, B_1, C_1, which are proportional to s^2/c^2, represent corrections to the hydrodynamic approximation $A \approx A_0$, $B \approx B_0$, $C \approx C_0$.

Thus the dispersion equation (12) takes on the form:

$$A_1 n^6 + (A_0 + B_1) n^4 + (B_0 + C_1) n^2 + C_0 = 0. \tag{24}$$

The three roots of this equation n_1^2, n_2^2 and n_3^2 determine, respectively, the indices of refraction of the ordinary, the extraordinary and the plasma waves.

Neglecting the terms in (24) proportional to s^2/c^2, we obtain the indices of refraction of the ordinary and the extraordinary waves in the hydrodynamic approximation[8]

$$n_{1,2}^2 = n_\pm^2 = (-B_0 \pm \sqrt{B_0^2 - 4A_0C_0})/2A_0. \tag{25}$$

Taking into account the thermal motion of the electrons we obtain

$$n_{1,2}^2 = (1 + \varepsilon_\pm) n_\pm^2, \tag{26}$$

$$\varepsilon_\pm = -(A_1 n_\pm^4 + B_1 n_\pm^2 + C_1)/(2A_0 n_\pm^2 + B_0),$$

$$|\varepsilon_\pm| \ll 1.$$

Since under the usual conditions $s^2/c^2 \ll 1$, the thermal corrections to $n_{1,2}^2$ are very small.

If the absolute value of $A_0 = (\omega^2 - \omega_+^2) \times (\omega^2 - \omega_-^2)/\omega^2(\omega^2 - \omega_H^2)$ is small compared to unity, i.e., if ω^2 is close to ω_+^2 or to ω_-^2,

$$\omega_\pm^2 = {}^1/_2\,(\Omega^2 + \omega_H^2 \pm \sqrt{(\Omega^2 + \omega_H^2)^2 - 4\Omega^2\omega_H^2\cos^2\theta}), \tag{27}$$

then from (25) we obtain approximately,

$$n_1^2 = -C_0/B_0, \quad n_2^2 = -B_0/A_0. \tag{28}$$

Since $B_0(\omega) > 0$ for $\omega^2 \approx \omega_\pm^2$, then $n_2^2 \to +\infty$ as $\omega^2 \to \omega_+^2$ (or ω_-^2) from the direction $\omega^2 < \omega_+^2$ (or $\omega^2 < \omega_-^2$) and $n_2^2 \to -\infty$ as $\omega^2 \to \omega_+^2$ (or ω_-^2) from the direction $\omega^2 > \omega_+^2$ (or $\omega^2 > \omega_-^2$). However, for very small values of A_0 the expression (28) for n_2^2 no longer holds, since it was obtained under the condition

$$s^2/c^2 \ll |A_0(\omega)| \ll 1. \tag{29}$$

The condition (29) means that the terms which were discarded in (24) in order to obtain (28) are small compared to the terms which were retained.

For the index of refraction of the plasma wave we obtain by the condition (29)

$$n_3^2 = -A_0/A_1. \tag{30}$$

Expression (30) coincides with the expression for n_3^2, which was obtained by Gershman.[6] If $|A_0| \gtrsim 1$, then the third solution of the cubic equation (24) for n^2 does not satisfy the condition $sn/c \ll 1$, which must hold if the expressions (20) for ϵ_{ik} are to be valid.

Let us now find the solution of (24) for $|A_0| \ll 1$. The index of refraction for the ordinary wave is determined as before by (29). We obtain the indices of refraction for the extraordinary and the plasma waves by assuming that for $|A_0| \ll 1$ $n_{2,3}^2 \gg 1$. Retaining the largest terms in (24) we obtain

$$n_{2,3}^2 = A_0\left(-1 \pm \sqrt{1 - \frac{4s^2A_1B_0}{c^2A_0^2}}\right)\Big/ 2A_1 s^2/c^2. \tag{31}$$

In the limiting case $|A_0| \gg s/c$ (31) leads to (28) and (30). For $|A_0| \ll s/c$ we obtain* from (31)

$$n^2 = (c/s)\sqrt{-B_0/A_1}. \tag{32}$$

The second solution given by (31) will be negative in this case.

4. THE CASE OF RESONANCE

Let us now examine the case of resonance $\omega \approx \omega_H$. Assuming that in the integrals occurring in the expressions for ϵ_{ik} in (10)

$$|\lambda| \ll 1, \quad |z_1| = \sqrt{{}^3/_2}\,|\omega - \omega_H|/k_z s \ll 1$$

and consequently that

$$\int\limits_C \frac{e^{-y^2}}{z_1 - y} dy \approx -i\pi,$$

we obtain, neglecting terms proportional to $(ka)^2$ or to z_1,

---

* For $\theta = \pi/2$, (32) gives Gershman's result[6].

$$\varepsilon_{11} = \varepsilon_{22} = 1 - \frac{v}{4} + i\sqrt{\frac{3\pi}{8}}\,\frac{v}{(ns/c)\cos\theta}; \tag{33}$$

$$\varepsilon_{12} = -\varepsilon_{21} = i\,\frac{v}{4}$$

$$-\sqrt{\frac{3\pi}{8}}\,\frac{v}{(ns/c)\cos\theta}; \quad \varepsilon_{33} = 1 - v;$$

$$\varepsilon_{23} = \varepsilon_{32} = \varepsilon_{13} = \varepsilon_{31} = 0.$$

We now substitute these expressions into the dispersion Eq. (12) under the assumption that n is small, and retain terms proportional to $c/s \gg 1$. We then obtain

$$n_{1,2}^2 = n_\pm^2 = \sin^{-2}\theta\,[1 + {}^1/_2\sin^2\theta - v \pm \tag{34}$$
$$\pm\sqrt{(1 + {}^1/_2\sin^2\theta - v)^2 - \sin^2\theta\,(1 - v)(2 - v)}].$$

In the next approximation we find

$$n_{1,2}^2 = (1 + \Delta_\pm)\,n_\pm^2, \tag{35}$$

$$\Delta_\pm = i\sqrt{\frac{8}{3\pi}}\,\frac{s\cos\theta}{cn_\pm v}$$
$$\times\frac{[1-({}^1/_4\sin^2\theta+\cos^2\theta)v]n_\pm^4-[(1-v)(1-{}^1/_4 v)(1+\cos^2\theta)+(1-{}^1/_2 v)\sin^2\theta]n_\pm^2+(1-v)(1-{}^1/_2 v)}{2\sin^2\theta\, n_\pm^2 + 2v - 2 - \sin^2\theta}. \tag{36}$$

Thus, the electromagnetic waves are damped when $\omega \approx \omega_H$. The order of magnitude of the damping coefficient is equal to s/c, i.e., it is appreciably larger than the usual thermal corrections to the indices of refraction of the ordinary and the extraordinary waves which are proportional to s^2/c^2.

As the angle θ is decreased $n_1 \approx n_+$ increases ($n_+^2 \to 2(1-v)\theta^{-2}$]; however, for small values of θ one cannot use expression (34) for n_1^2, since it was obtained under the condition

$$\theta \gg v^{-1/4}\sqrt{|1-v|}\,(s/c)^{1/4},$$

which means that the terms in the dispersion equation which are proportional to c/s are the largest ones.

5. LONGITUDINAL PLASMA OSCILLATIONS

Let us now consider in greater detail the problem of the longitudinal plasma oscillations. As is well known[4], in the presence of a magnetic field the electromagnetic waves in a plasma cannot be separated into strictly longitudinal and transverse ones. However, in the limiting case $n \gg 1$ we can distinguish a longitudinal plasma wave, the dispersion equation for which may be written approximately in the form: $A(\omega, \mathbf{k}) = 0$.

Substituting (9) into (13) we reduce this equation to the form*:

$$k^2a^2 + 1 - i\,\frac{\omega'}{\omega_H}\int\frac{f_0}{n_0}\,e^{i\alpha\sin\vartheta + i\beta\vartheta}\left\{\int_0^{\vartheta} e^{-i\alpha\sin\psi - i\beta\psi}d\psi\right. \tag{37}$$
$$\left. + (1 - e^{2\pi i\beta})^{-1}\int_0^{2\pi} e^{i\alpha\sin\psi + i\beta\psi}d\psi\right\}d\mathbf{v} = 0,$$

where

$$\alpha = k_x v_z/\omega_H, \quad \beta = (k_z v_z - \omega')/\omega_H, \quad \omega' = \omega - i\gamma,$$

ω' is the complex frequency [we have replaced ω by ω' in (9)]. In the future we shall take the propagation vector $\mathbf{k}$ to be real. Equation (37) then determines the frequency ω and the damping γ as functions of $\mathbf{k}$.

Making use of relation (10), and carrying out the integration over the angles in (37), and then over u, we finally obtain[10]

$$k^2a^2 + 1 - e^{-\mu}\sum_{n=-\infty}^{\infty} I_{|n|}(\mu)\,\frac{z_0}{\sqrt{\pi}}\int_C \frac{e^{-y^2}}{z_n - y}\,dy = 0,$$
$$\mu = k_x^2 s^2/3\omega_H^2, \quad z_n = \sqrt{{}^3/_2}\cdot(\omega' - n\omega_H)/k_z s, \tag{38}$$

where $I_n(\mu)$ is the modified Bessel function.

For $k_x = 0$ ($\mu = 0$) the dispersion equation (38) has the same form as in the absence of magnetic field[5], i.e., the magnetic field has no effect on plasma oscillations being propagated parallel to it.

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* We note that (37) may be obtained directly by starting with the kinetic equation and with the equation $\operatorname{div}\mathbf{E} = 4\pi e\int f d\mathbf{v}$.

In the case of "low" temperatures of the plasma, when $\mu \ll 1$ by expanding the functions $I_n(\mu)$ and $e^{-\mu}$ in powers of μ and by using the asymptotic expansion of the integral

$$\frac{1}{\sqrt{\pi}}\int\limits_C \frac{e^{-y^2}}{z-y}dy \approx \frac{1}{z}\left(1+\frac{1}{2z^2}+\frac{3}{4z^4}+\ldots\right) - i\sqrt{\pi}e^{-z^2}, \ (|z| \gg 1, |\mathrm{Im}\, z| \ll 1), \tag{39}$$

we obtain

$$\begin{aligned} k^2a^2 &- \left(\frac{1}{2z_0^2}+\frac{3}{4z_0^4}+\ldots\right) \\ &- \mu\left[\frac{z_0}{2}\left(\frac{1}{z_1}+\frac{1}{z_{-1}}+\frac{1}{2z_1^3}+\frac{1}{2z_{-1}^3}\right)-1-\frac{1}{2z_0^2}\right] \\ &- \mu^2\left[\frac{z_0}{8}\left(\frac{1}{z_2}+\frac{1}{z_{-2}}-\frac{4}{z_1}-\frac{4}{z_{-1}}\right)+\frac{3}{4}\right] \\ &+ \ldots + i\sqrt{\pi}z_0\Big[\left(1-\mu+\frac{3}{4}\mu^2\right)e^{-z_0^2} \\ &+ \frac{\mu}{2}(1-\mu)\left(e^{-z_1^2}+e^{-z_{-1}^2}\right) \\ &+ \frac{\mu^2}{8}\left(e^{-z_2^2}+e^{-z_{-2}^2}\right)+\ldots\Big] = 0. \end{aligned} \tag{40}$$

If we neglect the thermal motion of the electrons then (40) reduces to the dispersion equation of the hydrodynamic approximation

$$1 - \frac{\Omega^2}{\omega^2}\cos^2\theta - \frac{\Omega^2}{\omega^2-\omega_H^2}\sin^2\theta = 0,$$

from which we obtain the characteristic frequency of plasma oscillations in the hydrodynamic approximation[3]:

$$\omega^2 = \omega_\pm^2 = {}^1/_2(\Omega^2+\omega_H^2) \pm {}^1/_2\sqrt{(\Omega^2+\omega_H^2)^2 - 4\Omega^2\omega_H^2\cos^2\theta}. \tag{41}$$

Taking into account that $ka \ll 1$, we look for the solution of the dispersion equation (40) in the form

$$\omega_{1,2}^2 = (1+\varepsilon_\pm)\,\omega_\pm^2, \quad |\varepsilon_\pm| \ll 1. \tag{42}$$

For the corrections $\epsilon_\pm$ to the characteristic frequencies $\omega_{1,2}$, we obtain the expression

$$\begin{aligned} \varepsilon_\pm = \frac{k^2s^2}{\omega_\pm^2}\, &\frac{v_\pm}{[1 + v_\pm u_\pm (1-u_\pm)^{-2}\sin^2\theta} \\ \times &\left\{\cos^4\theta + \frac{\left(2-u_\pm+\frac{1}{3}u_\pm^2\right)\cos^2\theta\sin^2\theta}{(1-u_\pm)^3} + \frac{\sin^4\theta}{(1-u_\pm)(1-4u_\pm)}\right\}, \\ &v_\pm = \Omega^2/\omega_\pm^2, \quad u_\pm = \omega_H^2/\omega_\pm^2. \end{aligned} \tag{43}$$

Thus, in the case of low plasma temperatures ($\omega_H \gg ks$, "strong" magnetic field), there exist two characteristic frequencies of plasma oscillation, which are determined by Eqs. (41)-(43). We obtain the damping which corresponds to these frequencies by taking into account in Eq. (40) terms which are exponentially small:

$$\begin{aligned} \gamma_{1,2} = \sqrt{\frac{\pi}{8}}\, &\frac{\omega_\pm^2}{\Omega(ka)^3\cos\theta}\,\frac{1}{[1+\Omega^2\omega_H^2\sin^2\theta/(\omega_\pm^2-\omega_H^2)^2]} \\ &\times\Big\{\exp\{-\omega_{1,2}^2/2\Omega^2k^2a^2\cos^2\theta\} + \frac{k^2a^2\Omega^2}{2\omega_H^2} \\ &\times\sin^2\theta\,[\exp\{-(\omega_{1,2}-\omega_H)^2/2\Omega^2k^2a^2\cos^2\theta\} \\ &+ \exp\{-(\omega_{1,2}+\omega_H)^2/2\Omega^2k^2a^2\cos^2\theta\}] + \ldots\Big\}. \end{aligned} \tag{44}$$

Expression (42) was obtained under the condition $|z_n| \gg 1$ ($|\epsilon_+| \ll 1$). If $\theta \to 0$, then $\omega_1 \approx \omega_+ \to \omega_H$ (for $\Omega < \omega_H$), and $\omega_2 \approx \omega_- \to \omega_H$ (for $\Omega > \omega_H$). In this case the inequality $|z_1| \gg 1$ is not fulfilled, and the expressions (42) no longer hold. From the condition $|z_1| \gg 1$ (or from the condition $|\epsilon_\pm| \ll 1$), we find that the applicability of the Eq. (42) for ω_1 with $\omega_H > \Omega$ and for ω_2 with $\omega_H < \Omega$ is restricted by the condition

$$\theta \gg \sqrt{2\sqrt{2}ka\,|\Omega^2-\omega_H^2|/\Omega\omega_H}. \tag{45}$$

A unique solution $\omega \approx \Omega$ exists, as may be seen from the exact dispersion equation (38) for $\theta = 0$ and $ka \ll 1$.

If $\omega_H < \Omega$, then for certain ω_H and Ω there exists an angle $\theta = \theta_m$, for which the frequency

ω_1, determined by (41), turns out to be an integral multiple of ω_H:

$$\omega_1 \approx m\omega_H \quad (m = 2,\ 3,\ \ldots). \tag{46}$$

However, the dispersion equation (40) was itself obtained under the assumption $|z_n| \gg 1$. Therefore, one cannot use expression (42) at angles close to θ_m, if (46) holds for these angles. In order to obtain a dispersion equation which is valid for $\theta \approx \theta_m$, one should retain the mth integral in (38), and one should use for the other integrals the asymptotic expansion (39), as was done earlier:

$$k^2a^2 - \frac{1}{2z_0^2} - \mu\left[\frac{z_0}{2}\left(\frac{1}{z_1} + \frac{1}{z_{-1}}\right) - 1\right]. \tag{47}$$

$$+ \ldots - \left(\frac{\mu}{2}\right)^m \frac{z_0}{m!\,\sqrt{\pi}} \int\limits_C \frac{e^{-y^2}}{z_m - y}\,dy = 0.$$

If θ_m is not close to $\pi/2$, then assuming that

$$|z_m| \ll 1, \quad \int\limits_C \frac{e^{-y^2}}{z_m - y}\,dy \simeq -i\pi,$$

we find $\omega' = m\omega_H - i\gamma_m$, where

$$\gamma_m = \frac{\sqrt{\pi}\, m^4 \sin^{2m}\theta}{2^{m+3/2}3^{m-3/2}m!\cos^3\theta\,(1 + m^4(m^2-1)^{-2}\operatorname{tg}^2\theta)} \tag{48}$$

$$\times \left(\frac{ks}{\omega_H}\right)^{2m-4} ks$$

$$(m = 2,\ 3,\ \ldots).$$

Thus the waves with frequencies which are integral multiples of ω_H are strongly damped for θ not close to $\pi/2$. The damping coefficient γ_m is proportional to $(ks/\omega_H)^{2m-4}$ and decreases as m increases. For $m = 2$ the damping exceeds in order of magnitude by a factor $(ka)^{-1}$ the usual thermal corrections to the frequency.

If $\theta_m \approx \pi/2$, then assuming that

$$\int\limits_C \frac{e^{-y^2}}{z_m - y}\,dy \approx \frac{\sqrt{\pi}}{z_m}, \quad |z_m| \gg 1,$$

we obtain

$$\omega = m\omega_H \pm \varepsilon_m,\ \varepsilon_m \tag{49}$$

$$= (m^2 - 1)(2^{m+1}3^{m-1}m!)^{-1/2}(ks/\omega_H)^{m-2}ks.$$

Thus, for $\theta = \pi/2$ and for a given magnetic field, longitudinal waves with frequencies in the range $m\omega_H - \epsilon_m < \omega < m\omega_H + \epsilon_m$ cannot be propagated in the plasma. The width of the "gap" $2\epsilon_m$ decreases as m increases. Gross[4] has pointed out the existence of such "gaps" and has computed the value of ϵ_2.

Finally, if $\omega_H \approx \Omega$, Eqs. (42) are not applicable for small θ, since in such a case the condition $|z_1| \gg 1$ is not fulfilled. In this case the exact dispersion equation (38) takes on the form

$$1 - \frac{\Omega^2}{\omega^2} - \frac{\theta^2\omega}{2\sqrt{2}\,ka\Omega\sqrt{\pi}} \int\limits_C \frac{e^{-y^2}}{z_1 - y}\,dy = 0. \tag{50}$$

Assuming that $|z_1| \ll 1$ we find that

$$\omega = \Omega \approx \omega_H; \tag{51}$$

$$\gamma = {}^1/_4\sqrt{\pi/2}\,(\theta^2/ka)\,\Omega; \quad \theta \ll 2\sqrt{ka},$$

i.e., for $\omega_H \approx \Omega$ the plasma wave of frequency $\omega = \Omega \approx \omega_H$ is strongly damped.

In the case of a weak magnetic field ($\omega_H \ll \Omega$) Eq. (37) can be brought to the form

$$k^2a^2 - \frac{1}{\sqrt{\pi}} \int\limits_C \frac{ye^{-y^2}}{z - y}\,dy \tag{52}$$

$$- \frac{\omega_H^2 \sin^2\theta z^3}{4\Omega^2\sqrt{\pi}} \int\limits_C \frac{ye^{-y^2}}{(z - y)^4}\,dy - \ldots = 0, \quad z = \sqrt{\frac{3}{2}}\,\frac{\omega'}{ks}.$$

by means of integration by parts.

Solving this equation by successive approximations ($|z| \gg 1$), we find the frequency ω and the damping γ of the plasma oscillations in a weak magnetic field

$$\omega = \Omega + k^2s^2/2\Omega + \omega_H^2\sin^2\theta/2\Omega, \tag{53}$$

$$\gamma = \sqrt{\frac{\pi}{8}}\,\frac{\Omega}{(ka)^3}\left(1 + \frac{\omega_H^2\sin^2\theta}{4!(ka)^2\,\Omega^2}\right)e^{-\omega^2/2k^2a^2\Omega^2}. \tag{54}$$

Expressions (53) and (54) agree with the results obtained by Gordeev[5]. For $\omega_H = 0$ we obtain from this Vlasov's formula for the frequency[1] and Landau's formula for the damping*

$$\gamma = \sqrt{\pi/8}\,\Omega\,(ka)^{-3} e^{-1/2k^2a^2} e^{-3/2}.$$

In conclusion we express our sincere thanks to Prof. A. I. Akhiezer for his attention, for his assistance, and for the detailed discussion of the results of this work.

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* We note that in the expression for γ obtained by Landau[2] the factor $e^{-3/2}$ is missing.

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Investigation of the $Be^9(dn)B^{10}$ Nuclear Reaction

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An investigation is carried out on the reaction between the nucleus of beryllium and a deuteron in which the latter is captured by the nucleus and the unpaired neutron is ejected. The effective cross section of the process and the angular distribution of the freed neutrons were found. The comparison of the angular distribution with experimental data results in a satisfactory agreement for small angles up to 70°.

1. THE model of the Be^9 nucleus, according to which the unpaired neutron moves in the field of the nuclear remainder Be^8, was applied by many investigators to the problem of the electron and photoelectric disintegration of this nucleus[1-3]. The success of this model is determined first by the weak binding of the unpaired neutron in the Be^9 nucleus, considerably smaller than the mean binding energy per particle , and second, by the relatively long life of the Be^8 nuclear remainder relative to the decay into two α-particles. The current research is dedicated to the investigation of the $Be^9(dn)B^{10}$ reaction on the basis of this model[1].

It is customarily assumed that the (dn) reaction can proceed by the formation of a compound nucleus and stripping of a proton by a nucleus from a deuteron passing nearby. Calculations on the basis of the compound nucleus model are often not feasible in actual cases because the line widths of the corresponding processes are unknown; therefore, most of the theoretical investigations of the (dn) reaction are made from the point of view of the stripping process. The corresponding angular distributions of neutrons are then determined on the basis of Butler's theory[4]. When there is no agreement between this theory and experiment it is pointed out that in such cases the reaction does not proceed by stripping, but by the formation of a compound nucleus, which then undergoes various cascade transitions.

The probability of the stripping process is more or less apparent because the deuteron binding energy is considerably smaller than the binding energy of each particle in the nucleus with which the deuteron collides. However, this specific condition does not hold for the case of the beryllium nucleus. As is known, the binding energy of the unpaired neutron in the Be^9 nucleus amounts to only 1.66 mev, while the binding energy of the deuteron equals 2.23 mev. The low binding energy of the neutron in the Be^9 nucleus points to the assumption that the mean distance of the neutron from the nuclear remainder is relatively large. It is natural, therefore, to assume that the deuteron flying nearby will interact primarily with the unpaired neutron only, without causing any observable excitations in the nuclear remainder Be^8. The method proposed by us for calculating the (dn) reaction in the Be^9 nucleus is in a sense intermediary between the stripping and the compound nucleus methods. We are assuming that in the (dn) reaction the Be^9 nucleus interacts not only with one of the particles in the deuteron, as allowed by the stripping theory, but with both particles, the neutron and proton, whereupon the deuteron does not interact with the entire nucleus but only with its unpaired neutron.

Since the binding energy of the neutron in the Be^9 nucleus is smaller than the deuteron binding energy, there is a definite probability that before the interacting deuteron decays into a proton and a neutron, the unpaired neutron in the Be^9 nucleus will be ejected out of it. In this paper we are considering this very case.

2. It is known that the unpaired neutron in the Be^9 nucleus is found in the *p*-state. It follows from the shell model that the proton and the neutron in the B^{10} nucleus are also found in the *p*-state. Furthermore, it is not difficult to see that the binding energy of the last proton in the B^{10} nucleus is equal to

$$\Delta\varepsilon = \varepsilon(B^{10}) - \varepsilon(Be^9) = 6.62 \text{ mev.} \tag{1}$$

Furthermore, taking into account that the binding energy of the deuteron $\epsilon_d = 2.23$ mev, we obtain for the energy liberated in the reaction under consideration (on the assumption that B^{10} is found in the ground state)

$$Q = (Be^9 d,\, B^{10} n) = \Delta\varepsilon - \varepsilon_d = 4.29 \text{ mev.} \tag{2}$$

3. For the interaction energy between the deuteron and the Be^9 nucleus we take the expression

$$V = g\,[\delta(\mathbf{r}_1 - \mathbf{r}) + \delta(\mathbf{r}_2 - \mathbf{r})], \tag{3}$$

where $\mathbf{r}_1$ and $\mathbf{r}_2$ are the radius vectors of the deuteron's neutron and proton; $\mathbf{r}$ is the radius vector of the neutron in the Be^9 nucleus; $g = (4\pi\hbar/M)^2 a$, where M is the mass of the proton and neutron, and a is the scattering length associated with the effective cross section of neutron scattering on free protons by the formula

$$\sigma = 4\pi a^2.$$

Equation (1) for the interaction energy is correct so long as the wavelength of incident particles is greater than the effective radius of nuclear forces; therefore, we restrict ourselves to those incident deuterons whose energies do not exceed 1-2 mev. The wave functions for the initial and final states are

$$\psi_i = v_d^{-1/2} e^{i\mathbf{k}(\mathbf{r}_1+\mathbf{r}_2)/2}\, \Phi_d(|\mathbf{r}_1 - \mathbf{r}_2|)\, \psi_1(\mathbf{r}), \tag{4}$$
$$\psi_f = (2\pi\hbar)^{3/2} e^{i\mathbf{k}'\mathbf{r}}\, \psi_2(\mathbf{r}_2)\, \psi_3(\mathbf{r}_1),$$

where $\mathbf{k}$ is the wave vector of the incident deuteron, $\mathbf{k}'$ is the wave vector of the ejected neutron, v_d is the velocity of the incident deuteron, $\psi_1(\mathbf{r})$ is the wave function of the bound neutron in the Be^9 nucleus, ψ_2 and ψ_3 are wave functions of the proton and neutron in the B^{10} nucleus, Φ_d is the wave function of the deuteron's inner state.

Inasmuch as we are assuming that in the reaction under consideration the ejected neutron belongs to the Be^9 nucleus, and the incident neutron, decaying into a neutron and proton, is bound in the nucleus forming B^{10}, then we can consider that the proton-neutron distance is on the average the same in both the deuteron and in the B^{10} nucleus and, therefore, in calculating the matrix element in the deuteron function $\Phi_d(|\mathbf{r}_1 - \mathbf{r}_2|)$ the value $|\mathbf{r}_1 - \mathbf{r}_2|$ can be equated to the more probable neutron-proton distance in the deuteron, equal to

$$r_d = \hbar / \sqrt{\mu\varepsilon_d}.$$

4. As noted above, the unpaired neutron in the Be^9 nucleus, as well as the proton in the B^{10}, are found in the p-state. Therefore, in the matrix element of transition ψ_1, ψ_2 and ψ_3 are replaced by the corresponding equations for p-waves, with the radial wave functions determined from the assumption that the graphs of the neutron-proton interaction in the B^{10} nucleus and for the interaction between the neutron in the Be^9 nucleus and the Be^{10} nuclear remainder have the shape of a square well.

If one transforms into spherical waves the plane waves $e^{ikr/2}$ and $e^{-ik'r}$ corresponding to the incident deuterons and the neutrons ejected from the nucleus, and restricts the transformation to the values of orbital momentum $l = 0$ and $l = 1$ (so as to exclude as far as possible the stripping effect) then one finally gets after calculation of the differential section for the process under study,

$$d\sigma = 512\pi^6 (\mu / M)^2 a^2 \sqrt{{}^2/_5 (1 + Q / E_d)} \tag{5}$$

$$\times (r_0 / r_d)^3 e^{-2} (1 + r_0' / r_d)^2$$

$$\times \frac{(k_1 r_0)^2}{(kr_0)^2 (k' r_0)^2 (k_3 r_0)^2} \frac{F_1^2 F_2^2}{g_1 g_2 g_3} \sum_{i=0}^{6} \beta_i \cos^i \vartheta \, d\Omega_n,$$

where μ is the effective mass of the neutron and the proton relative to the nuclear remainder Be^8, E_d is the energy of the incident deuteron, r_0 is the radius of the potential well for the Be^9 and Be^{10} nuclei, r_0' is the neutron-proton interaction radius,

$$k = (6 / \hbar) \sqrt{M E_d / 11}, \tag{6}$$

$$k' = \hbar^{-1} \sqrt{2\mu (E_d + Q)}, \quad \beta_i = a_i / B_{00}^2;$$

$$F_1 = \int_0^{k_2 r_0} J_{1/4}\left(\frac{k}{2k_2} x\right) J_{1/2}\left(\frac{k'}{k_2} x\right) J_{3/2}\left(\frac{k_1}{k_2} x\right) J_{3/2}(x)\, dx \tag{7}$$

$$+ \frac{2}{\pi} \left(\frac{k_1}{\alpha_1}\right)^{1/2} \left(\frac{k_2}{\alpha_1}\right)^{1/2}$$

$$\times \left(\frac{k_2}{\alpha_2}\right)^2 \int_{k_2 r_0}^{\infty} \frac{(1 + \alpha_1 x / k_2)(1 + \alpha_2 x / k_2)}{x^3} J_{1/2}\left(\frac{k}{2k_2} x\right)$$

$$\times J_{1/2}(x)\, e^{-(\alpha_1 + \alpha_2)(x - k_2 r_0)/k_2} dx;$$

$$F_2 = \int_0^{k_3 r_0} x J_{3/2}\left(\frac{k}{2k_3} x\right) J_{3/2}(x)\, dx \tag{8}$$

$$- \sqrt{\frac{2}{\pi}} \left(\frac{k_2}{\alpha_3}\right)^2 \sin k_3 r_0$$

$$\times \int_{k_3 r_0}^{\infty} \frac{(1 + \alpha_1 x / k_3) J_{3/2}(kx / 2k_3)}{\sqrt{x}} e^{-\alpha_3 (x - k_3 r_0)/k_3} dx;$$

$$g_\mu = k_\mu r_0 \tag{9}$$

$$+ \left[(2 + \alpha_\mu r_0) \frac{k_\mu^4}{\alpha_\mu^4} + (1 + \alpha_\mu r_0) \frac{k_\mu^2}{\alpha_\mu^2} - 1 \right] \frac{\sin^2 k_\mu r_0}{k_\mu r_0};$$

$$(\mu = 1,\ 2,\ 3);$$

$$a_0 = (3/16) B_{00}^2 + (81/200) B_{11}^2 + (3/8) B_{02}^2$$

$$+ (3927/9800) B_{13}^2 + (567/700) B_{11} B_{13};$$

$$a_1 = (15/16) B_{00} B_{11} - (9/10) B_{02} B_{11}$$

$$+ (27/20) B_{02} B_{13};$$

$$a_2 = (513/400) B_{11}^2 + ({}^3/_8) B_{02}^2 + (5/16) B_{02}^2$$

$$+ (14/23) B_{13}^2 - (567/175) B_{11} B_{13};$$

$$a_3 = (3/4) B_{02} B_{11} + (189/112) B_{02} B_{13};$$

$$a_4 = (65/96) B_{02}^2; \quad a_5 = -(8663/4200) B_{02} B_{13};$$

$$a_6 = (21/196) B_{13}^2,$$

with

$$B_{ll'} = \int f_l\left(\frac{kr}{2}\right) f_{l'}(k'r) R_1(r) R_2(r) r^2 dr; \tag{10}$$

$$k_i = \hbar^{-1} \sqrt{2\mu (V_i - \varepsilon_i)}, \tag{11}$$

$$\alpha_i = \sqrt{2\mu\varepsilon_i} / \hbar, \quad (i = 1,\ 2,\ 3).$$

In the last equations ϵ_1 represents the binding energy of the neutron in the Be^9 nucleus, ϵ_2 and ϵ_3 are the binding energies of the corresponding proton and neutron in the B^{10} nucleus, and V_1, V_2 and V_3 are the corresponding values of the potential well depth, with the assumption that the radius of the well r_0 has the same value for the Be^9 and B^{10} nuclei.

In Eq. (10), $R_1(r)$ and $R_2(r)$ are the radial wave functions of the p-state, i.e.,

$$R_i(r) = b_i (k_i r)^{-1/2} J_{3/2}(k_i r) \quad \text{for} \quad r \leqslant r_0, \tag{12}$$

$$R_i(r) = c_i \alpha_i^{-1/2} (1 + \alpha_i r) r^{-2} e^{-\alpha(r - r_0)} \quad \text{for} \quad r \geqslant r_0,$$

$$f_n(z) = (\pi/2z)^{1/2} J_{n+1/2}(z), \tag{13}$$

where $J_m(n)$ is a Bessel function of order m. The coefficients b_i and c_i are determined from the conditions of continuity and normalization, namely,

$$c_i = -\sqrt{2/\pi}\, b_i \alpha_i^{-3/2} \sin k_i r_0; \quad b_i^2 = \pi k_i^2 / g_i, \tag{14}$$

and the values of k_1, k_2 and k_3 are the roots of the transcendental equation

$$\operatorname{tg} k_i r_0 = \frac{k_i r_0}{1 + (1 + \alpha_i r_0)\, k_i^2 / \alpha_i^2}\,. \tag{15}$$

The radius of the potential well for the interaction of Be^9 is well known from the investigations on electron and photoelectric disintegrations of beryllium. It is equal to 5×10^{-13} cm. Taking this value for r_0 and requiring that the well depth be in the 10-30 mev range, one can uniquely determine the coefficients k_i from Eq. (15). As a result of solving Eq. (15), $k_1 r_0 = 3.35$, $k_2 r_0 = 3.63$, $k_3 r_0 = 3.64$, with $\alpha_1 r_0 = 1.33$, $\alpha_2 r_0 = 2.66$ and $\alpha_3 r_0 = 3.02$.

After integrating Eq. (5) over the angles, we obtain the complete effective cross section for the reaction under study:

$$\begin{aligned} \sigma = {} & 4 \cdot 512 \cdot \pi^7 (\mu / M)^2 a^2 (r_0 / r_d)^2 \\ & \times e^{-2} (1 + r_0' / r_d)^2 \sqrt{2/5\,(1 + Q / E_d)} \\ & \times \frac{(k_1 r_0)^2}{(k r_0)^2 (k' r_0)^2 (k_3 r_0)^2} \\ & \times \frac{F_1^2 F_2^2}{g_1 g_2 g_3} \left(\beta_0 + \frac{1}{2} \beta_2 + \frac{2}{5} \beta_4 + \frac{3}{7} \beta_6 \right). \end{aligned} \tag{16}$$

5. In Ref. 5 the experimental investigation was carried out on the angular distribution of neutrons produced in the reaction $\mathrm{Be}^9(dn)\mathrm{B}^{10}$. The writers of this paper give the following function of neutron intensity versus the scattering angle for the case in which the B^{10} nucleus remains in the unexcited state:

$$f(\vartheta) = 0.87 - 0.28 \cos \vartheta + 0.42 \cos^2 \vartheta. \tag{17}$$

The graph of (17) is plotted with six experimental points and $E_d = 0.945$ mev and with the unit intensity taken as the intensity corresponding to $\vartheta = 0$. The second line of the Table gives values calculated from the empirical formula (17).

TABLE

ϑ	0	15°	40°	65°	90°	115°	140°	130°
$f(\vartheta)$	1	0.99	0.90	0.82	0.87	1.06	1.33	1.57
$f'(\vartheta)$	1	1.00	0.97	0.85	0.72	0.62	0.49	0.27

Using Eqs. (5)-(9) one can easily obtain an equation for the angular distribution of the neutrons for incident deuterons with energies of 0.945 mev.

If the intensity corresponding to $\vartheta = 0$ is taken as unity, then we have

$$\sum \beta_i \cos^i \vartheta = 0.347\, f'(\vartheta), \tag{18}$$

$$\begin{aligned} f'(\vartheta) = {} & 0.715 + 0.256 \cos \vartheta + 0.176 \cos^2 \vartheta \\ & + 0.121 \cos^3 \vartheta - 0.251 \cos^4 \vartheta - 0.017 \cos^5 \vartheta, \end{aligned}$$

with the $\cos^6 \vartheta$ term rejected because of the small value of the coefficient.

The comparison of the $f'(\vartheta)$ values for various scattering angles obtained by using Eq. (18) with the experimental values found from Eq. (17) shows that there is agreement between theory and experiment for small scattering angles in the range 0-70°. The discrepancies in numerical values are found for angles in the range 70-180°. The theoretical curve for small scattering angles on the whole describes correctly the dependence of the distribution function on the scattering angle.

In the third line of the Table are given the theoretical values of the neutron distribution as a function of the scattering angle. The discrepancy between theory and experiment for relatively large angles can be ascribed to the fact that the binding energy of the deuteron is comparable after all to the binding energy of the neutron in the Be^9 nucleus, so that those cases are also possible, not calculated by us, in which there is an ejection of the neutron belonging to the deuteron and not of the neutron from the Be^9 nucleus. For a complete description of neutron scattering with medium and large angles this fact requires the calculation of the exchange effect between the neutrons of the deuteron and those the Be^9 of nucleus. It is known that the calculation of the exchange results in an increase of the scattering probability for large angles.

As to the value of the complete effective cross section for the (dn) reaction with the Be^9 nucleus, there are no experimental data in the literature, and therefore, for the time being, Eq. (16) cannot be verified for the complete effective cross section.

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Dielectric Properties of Bismuth Titanate

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It is shown that the titanates of bismuth possess high dielectric constants ($\epsilon = 70$-120) and comparatively large positive temperature coefficients ($TC\epsilon = 130$-550×10^{-6}) which are evidently brought about by a combination of a favorable internal field and a high ionic polarizability. The formation process of one of the barium titanates is investigated, and the temperature and frequency dependencies of ϵ and $\tan\delta$ of various bismuth polytitanates have been measured.

THE titanates of metals of the second group of the periodic table have an elevated dielectric constant which increases upon increase in the atomic weight of the metal[1]. In this case, for nonpiezoelectric titanates, as well as for many other ceramic dielectrics, this rule holds: the higher the dielectric constant, the more negative the temperature coefficient $(1/\epsilon)(d\epsilon/dT)$[2]. For small dielectric constants the quantity $(1/\epsilon)(d\epsilon/dT)$ has a positive sign. For most dielectric constants, $(1/\epsilon)(d\epsilon/dT)$ has a comparatively large negative value. As shown earlier[3], the high dielectric constant of nonpiezoelectric titanates of barium and rutile (TiO_2) is brought about by a combination of high electronic polarization and a favorable polarization of the internal field produced by the ionic displacement. The ionic polarizability in this case is not large. The electronic polarizability does not depend on the temperature but the electronic polarizability per unit volume decreases with increase in temperature at the expense of a decrease in the number of polarized particles per unit volume in thermal expansion. The ionic polarizability increases with increase in temperature, since the elastic coupling is weakened. In dielectrics with high dielectric constant, the electronic polarizability appreciably exceeds the ionic; therefore, the dielectric constant decreases with increase in temperature $[(1/\epsilon)(d\epsilon/dT) < 0]$. The positive temperature coefficient of ϵ points to the predominance of the ionic polarizability.

In the present work, an attempt was made to realize, in a polycrystalline, nonpiezoelectric dielecrtic, such conditions under which a high dielectric constant would be combined with a comparatively large positive temperature coefficient. An appreciable ionic polarization takes place in glasses, the dielectric constant of which increases sufficiently rapidly with increase in temperature; therefore, the presence of glassy layers in a polycrystalline dielectric ought to facilitate an increase of ϵ with temperature. On the other hand, it is necessary that relaxation processes connected with inelastic dislocations of the ions not take place in this glassy layer and in defective places in the crystalline lattice, for this would, in turn, lead to greater loss. In this connection, the glassy layer must contain only heavy ions, possessing small mobility.

Starting from these considerations, we investigated the possibility of the formation of a combination of titanium dioxide and bismuth trioxide. Bismuth trioxide possesses weakly basic properties

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TABLE I

Molecular ratio of the components TiO_2/Bi_2O_3	Temperature of annealing, °C	ϵ, $f = 2$ mc, $t = 20°$	$\tan\delta$, $f = 2$ mc, $t = 20°$	$TC\epsilon \times 10^6$, $f = 2$ mc in the interval $t = 20$-$80°$	Molecular ratio of the components TiO_2/Bi_2O_3	Temperature of annealing, °C	ϵ, $f = 2$ mc, $t = 20°$	$\tan\delta$, $f = 2$ mc, $t = 20°$	$TC\epsilon \times 10^6$, $f = 2$ mc in the interval $t = 20$-$80°$
22.3 : 1	1240	92	0.0015	—540	2 : 1	1185	113	0.0054	+180
13.7 : 1	1220	89	0.0034	—390	1.7 : 1	1170	112	0.0040	+350
10.8 : 1	1215	86	0.0034	—140	1.45 : 1	1150	107	0.0023	+530
8.8 : 1	1215	80	0.0039	+460	1 : 1	1060	121	0.0024	+510
8.8 : 1	1245	91	0.0031	—530					
3 : 1	1220	68—86	0.0028—0.0040	from + 130 to + 590					

which appear both in solutions and at high temperature. Therefore, it can probably enter into a reaction at high temperature with titanium dioxide, which possesses weakly acidic properties, giving a compound of a type of bismuth polytitanate. Bismuth trioxide has a comparatively low melting point (about 900°). At this temperature, titanium dioxide undergoes a transition from the crystalline form of anatase into a crystalline modification of rutile and has an increased reaction capability. Therefore, we can expect that, at a temperature of about 900°, a reaction begins between TiO_2 present in the solid phase, and Bi_2O_3, present in the liquid phase. As a result of this reaction, polycrystals could be formed with appreciable glassy layers which contain the heavy bismuth ions and also defects in the lattice.

Thermographic analysis of the process of the formation of the titanates of bismuth confirms the assumptions made above. At $T = 900°$ an endothermic peak occurs, corresponding to the liquefaction of bismuth trioxide. A sharp endothermic peak in the differential temperature curve at 1230° shows the intense reaction which leads to the formation of one of the bismuth titanates corresponding to a given ratio of components. X-rays of the bismuth titanates also point to their chemical individuality.

Measurement of the dielectric constant and the power factor at high frequencies and for different polytitanates of bismuth shows that for an appreciable change in the ratio of components, the dielectric constant changes slightly and lies within the limits 70-120. The power factor has a higher value ($\tan\delta \approx 0.002$-0.006). The temperature coefficient of the dielectric constant, in agreement with theoretical assumptions, was shown to be positive for many bismuth titanates. The results of the measurements of ϵ, $\tan\delta$ and $TC\epsilon = (1/\epsilon)(d\epsilon/dT)$ are plotted in Table I for a number of bismuth titanates.

Figures 1 and 2 show curves for the dependence of ϵ and $\tan\delta$ on temperature and frequency for one of the bismuth titanates. It is seen that ϵ increases appreciably and almost linearly with temperature. For certain ratios of TiO_2/Bi_2O_3 (of those plotted in Table I) the temperature coefficient of ϵ undergoes a transition from positive to negative values. As specially arranged experiments have shown, in this region of concentration, $TC\epsilon$ always depends on the annealing history. The thermogram of Fig. 3 bears witness to this fact. It is evident therein that at a temperature of 1230° a sharp

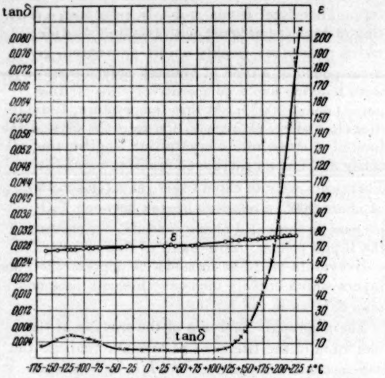

FIG. 1. Dependence of ε and tan δ on the temperature for $f = 700$ kc for one of the bismuth titanates.

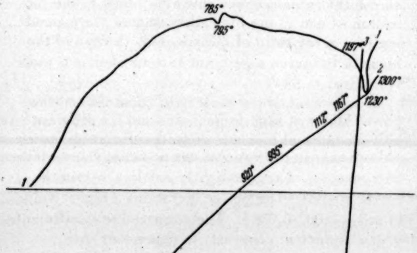

FIG. 3. Thermogram of the sintering of one of the bismuth titanates: 1 = difference in temperature of the specimen and the etalon; 2 = temperature of the furnace; 3 = shrinkage of the specimen.

change in the structure takes place. This is accompanied by an emission of heat (a phase change of first order) over a very narrow range of temperatures. If the temperature is 1-2° less than the transition temperature (1230°), then TCε has a large positive value; for the temperature 1230° and higher, TCε has a large negative value.

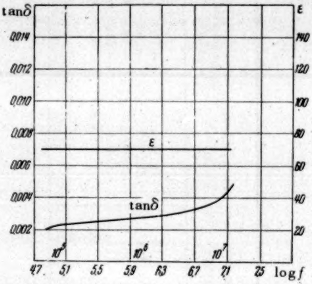

FIG. 2. Dependence of ε and tan δ on the frequency for one of the bismuth titanates.

In this transition, an appreciable increase takes place in the dielectric constant, and there is some decrease in the power factor. The experimental data which characterize the effect of the temperature annealing on the dielectric properties of one of the bismuth titanates are shown in Table II.

In conclusion, it is interesting to note that the discovery of such a sharp change in TCε and the dielectric constant is connected with the structure change which is noted in the thermogram (a sharp endothermic peak), but at the same time there is no effect on the Debye-gram. It is possible that in this transition a crystallization of the glass layer takes place. The problem of the change of the crystallographic structure in this case requires further x-ray investigation.

TABLE II. Change of dielectric properties of one of the bismuth titanates under the action of temperature annealing.

Temperature of annealing, °C	ε, $f = 2$ mc, $t = 20°$	tan δ, $f = 2$ mc, $t = 20°$	TCε×10^6 for $f = 2$ mc, in the interval $t = 20$-$80°$
1210	83	0.0032	+360
1220	78	0.0038	+360
1225	81	0.0045	+160
1230	83	0.0027	—430
1245	94	0.0022	—480

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On Quasiclassical Quantization

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A method of calculating the energy levels in a quasiclassical quantization is presented for the one-dimensional case. The value of the levels is obtained in the form of in expansion $\hbar$. Under certain assumptions on the form of the potential energy $U(x)$, this expansion can be obtained in a general form. Computations are carried out for a potential energy having a minimum and rising on either side of the minimum, i.e., of an oscillator type.

As is well known, in the quasiclassical method for the solution of the problems of quantum mechanics, the wave equation ψ is written in the form

$$\psi = e^{i\sigma/\hbar}. \tag{1}$$

Making this formal substitution in the Schrödinger equation

$$\Delta\psi + (2\mu/\hbar^2)(E - U) = 0, \tag{2}$$

we obtain an equation for the function σ:

$$(\nabla\sigma)^2 + (\hbar/i)\,\Delta\sigma = 2\mu\,(E - U). \tag{3}$$

The formal solution of Eq. (3) is written in the form of a power series in $\hbar$:

$$\sigma = \sigma_0 + (\hbar/i)\,\sigma_1 + (\hbar/i)^2\,\sigma_2 + \ldots \tag{4}$$

Substituting (4) in (3), we ultimately get, for the one-dimensional case,

$$\begin{gathered}\sigma_0' = p;\ \sigma_1' = -p'/2p;\ \sigma_2' = p''/4p^2 - 3p'^2/8p^3; \\ \sigma_3' = -p'''/8p^3 + 3p''p'/4p^4 - 3p'^3/4p^5; \\ \sigma_4' = p^{(IV)}/16p^4 - 5p'''p'/8p^5 \\ - 13p''^2/32p^5 + 99p''p'^2/32p^6 \\ -297p'^4/128p^7; \ldots,\end{gathered} \tag{5}$$

where $p = \sqrt{2\mu(E - U)}$ is the classical momentum.

For real p, the quantities $\sigma_0', \sigma_1', \ldots$ and $\sigma_0, \sigma_1, \ldots$ are real, and the quantity σ can be uniquely divided into two components which define the phase and modulus of the wave function:

$$\begin{aligned}\psi = {} & \exp\{\sigma_1 - \hbar^2\sigma_3 + \hbar^4\sigma_5 - \ldots\} \\ & \times \exp\{i(\overline{\sigma_0}/\hbar - \hbar\sigma_2 + \hbar^3\sigma_4 - \ldots)\}.\end{aligned} \tag{6}$$

Another linearly independent solution of the Schrödinger equation is obtained by substituting $i \to -i$ in Eq. (6). For imaginary p, all the expressions in the exponent are real.

Let $x = a$ be a turning point, i.e., $U(a) = E$. Let us find the phase of the wave function for $x > a$, considering that, in this region, $E > U(x)$, and in the region $x < a$, $E < U(x)$, and the modulus of the wave function decreases with decreasing x. Solving the Schrödinger equation exactly in the neighborhood of the turning point, where the potential energy can be approximated by a linear function of the coordinate x, and joining the exact solution with the quasiclassical one, we obtain an expression for the phase, as is usually done. The exact solution of the Schrödinger equation with a linear potential which satisfies the conditions set forth above has the form (except for a constant multiplier)

$$\psi = \begin{cases} \sqrt{|\xi|}\left[I_{-1/3}\left(\frac{2}{3}|\xi|^{3/2}\right) + I_{1/3}\left(\frac{2}{3}|\xi|^{3/2}\right)\right], & x < 0; \\ \sqrt{\xi}\left[J_{-1/3}\left(\frac{2}{3}\xi^{3/2}\right) + J_{1/3}\left(\frac{2}{3}\xi^{3/2}\right)\right], & x > 0, \end{cases}$$

$$\xi = \alpha x/\hbar^{3/2},\quad \alpha = \sqrt{2\mu(-\partial U/\partial x)_a}.$$

Its asymptotic expansion for $\hbar \to 0$ can be written for $x > 0$ in the form*

$$\frac{3}{\sqrt{\pi}}\frac{1}{\xi^{1/4}}\exp\left\{-\frac{5}{64}\frac{1}{\xi^3}+\frac{565}{2048}\frac{1}{\xi^6}-\dots\right\}\sin\left(\frac{2}{3}\xi^{3/2}+\frac{\pi}{4}-\frac{5}{48}\frac{1}{\xi^{3/2}}+\frac{1105}{9216}\frac{1}{\xi^{9/2}}-\dots\right). \tag{7}$$

The phase of $\frac{2}{3}\frac{\alpha x^{3/2}}{\hbar}+\frac{\pi}{4}-\frac{5}{48}\frac{\hbar}{\alpha x^{3/2}}+\frac{1105}{9216}\frac{\hbar^3}{\alpha^3 x^{9/2}}$ must be joined with the phase of the function

$$\exp\{\sigma_1-\hbar^2\sigma_3+\hbar^4\sigma_5-\dots\}\times\sin\left(\frac{\sigma_0}{\hbar}-\hbar\sigma_2+\hbar^3\sigma_4-\dots+\text{const}\right) \tag{8}$$

close to $x = a$, determining the unknown constant in this case.

At the point $x = a$, the momentum p vanishes; if x is considered as a complex variable, then for $p(x)$, the point $x = a$ is a branch point in which $p(x)$ is a double-valued function. The functions σ_0', σ_2', σ_4', ... are also double-valued from the branch point for $x = a$, as is evident from Eq. (5).

To obtain the functions σ_0, σ_2, σ_4, ..., it is appropriate to carry out the transformation from ordinary to contour integration, since the functions σ_2', σ_4', ... go to infinity for $x = a$. We make a cut in the complex plane x, going to the right from the point $x = a$; on the bottom side of the cut, let the square root take the positive sign, and on the upper side, the negative sign. Then the integral over x reduces to one-half the integral over the loop in which we go around from the point x on the upper side of the cut surrounding the point $x = a$ and proceed to the point x on the lower side of the cut.

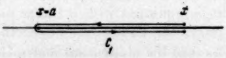

FIG. 1

For such a determination of the functions σ_0, σ_2, σ_4, ..., we have

$$\sigma_0 = {}^1/_2\int_{C_1}\sigma_0'\,dx; \tag{9}$$

$$\sigma_2 = {}^1/_2\int_{C_1}\sigma_2'\,dx; \quad \sigma_4 = {}^1/_2\int_{C_1}\sigma_4'\,dx, \dots$$

Comparing (6), (7) and (9), for x close to a, we obtain for the phase the value

$$\sigma_0/\hbar+\pi/4-\hbar\sigma_2+\hbar^3\sigma_4-\dots, \tag{10}$$

where the σ_0, σ_2, σ_4, ... are determined by Eqs. (9).

We now consider a form of the potential energy $U(x)$ in which there are two turning points, $x = a$, $x = b$, where for $a < x < b$, $U(x) < E$, and in the rest of the region, $U(x) > E$. The wave function which vanishes for $x < a$ has (for $x > a$) the form (it can be considered real), except for a constant multiplier,

$$\exp\left\{\sigma_1-\hbar^2\sigma_3+\hbar^4\sigma_5-\dots\right\}\times\sin\left(\frac{\sigma_0}{\hbar}+\frac{\pi}{4}-\hbar\sigma_2+\hbar^3\sigma_4-\dots\right), \tag{11}$$

where the σ_0, σ_2, σ_4, ... are determined by Eqs. (9). The wave function which vanishes for $x > b$ has (for $x < b$) the form

$$\exp\{\sigma_1-\hbar^2\sigma_3+\hbar^4\sigma_5-\dots\}\times\sin\left(\frac{s_0}{\hbar}+\frac{\pi}{4}-\hbar s_2+\hbar^3 s_4-\dots\right), \tag{12}$$

where the s_0, s_2, s_4, ... are determined by

$$s_0 = {}^1/_2\int_{C_2}\sigma_0'\,dx; \tag{13}$$

$$s_2 = {}^1/_2\int_{C_2}\sigma_2'\,dx; \quad s_4 = {}^1/_2\int_{C_2}\sigma_4'\,dx, \dots$$

The contour C_2 is a loop surrounding the point $x = b$ in a counter-clockwise direction, in which the cut is taken from the point $x = b$ to the left; on the lower side of the cut the square root is positive, on the upper side it is negative.

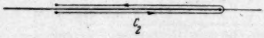

FIG. 2

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* The series in the exponent and in the sine argument are determined identically. Upon expansion of the exponent and the sine in powers of $\xi^{-3/2}$, we obtain an asymptotic expansion in a series in the usual form, keeping all successive terms of the expansion up to terms of that order which they have in the exponent and in the sine argument.

The wave function of the energy level must vanish for $x < a$ and $x > b$; the decompositions (11) and (12) represent one and the same function; therefore, the phases determined by Eqs.(11) and (12) must in total give an integral multiple of π, which leads to the condition

$$\oint_C \sigma_0' dx - \hbar^2 \oint_C \sigma_2' dx + \hbar^4 \oint_C \sigma_4' dx - \ldots = (n + 1/2)\, 2\pi\hbar, \qquad (14)$$

where the $\sigma_0', \sigma_2', \sigma_4', \ldots$ are determined by Eqs. (5), $n = 0, 1, 2, \ldots$, and the closed integration contour C surrounds the points $x = a$ and $x = b$ in the counter-clockwise direction. The condition (14) is the exact quantization rule of Bohr.

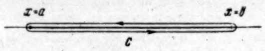

FIG. 3

The quantities $\sigma_0', \sigma_2', \sigma_4', \ldots$ entering into (14) are equal, according to Eq. (5), to the following:

$$\sigma_0' = \sqrt{2\mu(E-U)}; \qquad (15)$$

$$\begin{aligned}\sigma_2' = &- U''/8\sqrt{2\mu}\,(E-U)^{3/2} \\ &- 5U'^2/32\sqrt{2\mu}\,(E-U)^{5/2}; \\ \sigma_4' = &- U^{(IV)}/32\,(2\mu)^{3/2}(E-U)^{5/2} \\ &- 7U'''U'/32\,(2\mu)^{3/2}(E-U)^{7/2} \\ &- 19U''^2/128\,(2\mu)^{3/2}(E-U)^{7/2} \\ &- 221U''U'^2/256\,(2\mu)^{3/2}(E-U)^{9/2} \\ &- 1105U'^4/2048\,(2\mu)^{3/2}(E-U)^{11/2}, \ldots\end{aligned}$$

We carry out the integration in Eq. (14) in the general form under the supposition that the potential energy $U(x)$ at a certain point has a minimum, and at points $x = a$ and $x = b$, $U(a) = U(b) = E$. We locate the origin of the coordinates at $U(x)$ so that $U(0) = 0$, $U'(0) = 0$. We displace the contour of integration C which surrounds the points $x = a$ and $x = b$ in the complex plane x so that the condition $E < |U(x)|$ is satisfied on it, which permits us to expand the function (15) in a series in E. This can be accomplished in each case if the singular points of the function $\sqrt{E - U(x)}$ are sufficiently far from the points $x = a$, $x = b$.

Thus, over the entire path of integration, cut longitudinally by the method described above, we have

$$\sigma_0' = i\sqrt{2\mu U}\sqrt{1 - \frac{E}{U}} \qquad (16)$$

$$= i\sqrt{2\mu U}\left\{1 - \frac{1}{2}\frac{E}{U} - \frac{1}{2\cdot 4}\left(\frac{E}{U}\right)^2 - \frac{1\cdot 3}{2\cdot 4\cdot 6}\left(\frac{E}{U}\right)^3 - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 8}\left(\frac{E}{U}\right)^4 - \ldots\right\};$$

$$\sigma_2' = -\frac{iU''}{8\sqrt{2\mu}U^{3/2}}\left\{1 + \frac{3}{2}\frac{E}{U} + \frac{3\cdot 5}{2\cdot 4}\left(\frac{E}{U}\right)^2 + \ldots\right\} + \frac{5}{32}\frac{iU'^2}{\sqrt{2\mu}U^{5/2}}\left\{1 + \frac{5}{2}\frac{E}{U} + \frac{5\cdot 7}{2\cdot 4}\left(\frac{E}{U}\right)^2 + \ldots\right\};$$

$$\sigma_4' = \frac{iU^{(IV)}}{32\,(2\mu)^{3/2}U^{5/2}} - \frac{7}{32}\frac{iU'''U'}{(2\mu)^{3/2}U^{7/2}} - \frac{19}{128}\frac{iU''^2}{(2\mu)^{3/2}U^{7/2}} + \frac{221}{256}\frac{iU''U'^2}{(2\mu)^{3/2}U^{9/2}} - \frac{1105}{2048}\frac{iU'^4}{(2\mu)^{3/2}U^{11/2}} + \ldots,$$

Substituting (16) in (14), we obtain an equation for E, the solution of which is to be sought in the form of a series in $\hbar$:

$$E = \hbar E_1 + \hbar^2 E_2 + \hbar^3 E_3 + \ldots \qquad (17)$$

Then the equation takes the form

$$\begin{aligned} i\oint_C dx \Bigg\{ & \sqrt{2\mu U} + \hbar\left(-\frac{1}{2}\frac{\sqrt{2\mu}E_1}{U^{1/2}}\right) \\ & + \hbar^2\left(-\frac{\sqrt{2\mu}E_2}{2U^{1/2}} - \frac{\sqrt{2\mu}E_1^2}{8U^{3/2}} + \frac{U''}{8\sqrt{2\mu}U^{3/2}} - \frac{5U'}{32\sqrt{2\mu}U^{5/2}}\right) \\ & + \hbar^3\left(-\frac{\sqrt{2\mu}E_3}{2U^{1/2}} - \frac{\sqrt{2\mu}E_1E_2}{4U^{3/2}} - \frac{\sqrt{2\mu}E_1^3}{16U^{5/2}} + \frac{3U''E_1}{16\sqrt{2\mu}U^{5/2}} - \frac{25U'^2E_1}{64\sqrt{2\mu}U^{7/2}}\right) \\ & + \hbar^4\left(-\frac{\sqrt{2\mu}E_4}{2U^{1/2}} - \frac{\sqrt{2\mu}E_1E_3}{4U^{3/2}} - \frac{\sqrt{2\mu}E_2^2}{8U^{3/2}} - \frac{3\sqrt{2\mu}E_1^2E_2}{16U^{5/2}} - \frac{5\sqrt{2\mu}E_1^4}{128U^{7/2}} \right.\\ & + \frac{3U''E_2}{16\sqrt{2\mu}U^{5/2}} + \frac{15U''E_1^2}{64\sqrt{2\mu}U^{7/2}} - \frac{25U'^2E_2}{64\sqrt{2\mu}U^{7/2}} - \frac{175U'^2E_1^2}{256\sqrt{2\mu}U^{9/2}} \\ & + \frac{U^{(IV)}}{32\,(2\mu)^{3/2}U^{5/2}} - \frac{7U'''U'}{32\,(2\mu)^{3/2}U^{7/2}} - \frac{19U''^2}{128\,(2\mu)^{3/2}U^{7/2}} + \frac{221U''U'^2}{256\,(2\mu)^{3/2}U^{9/2}} \\ & \left. - \frac{1105U'^4}{2048\,(2\mu)^{3/2}U^{11/2}}\right) + \ldots\Bigg\} = \left(n + \frac{1}{2}\right)2\pi\hbar. \end{aligned}$$

Equating terms with equal powers of $\hbar$, we can, term by term, find the coefficients of the series (17). The term without $\hbar$ is identically equal to zero, since the integrand $+\sqrt{U(x)}$ is a single-valued function which has no singularities inside the contour C [we recall that $U(0) = 0$, $U'(0) = 0$].

The terms for $\hbar$ give

$$-i\frac{\sqrt{2\mu E_1}}{2}\oint_C \frac{dx}{\sqrt{U}} = \left(n + \frac{1}{2}\right)2\pi.$$

The function under the integral, $1/+\sqrt{U(x)}$, is a single-valued function having one pole within the contour C (at $x = 0$) with residue

$$\sqrt{2/U''(0)},$$

therefore,

$$E_1 = \sqrt{U''(0)/\mu}\,(n + {}^1/_2). \tag{18}$$

Equating to zero the terms in $\hbar^2$, we take it into consideration that each of the components in the integrand is single-valued inside the contour C and has a single pole at $x = 0$ with a corresponding residue. Computing these residues and using E_1 from (18), we obtain

$$E_2 = \frac{1}{\mu}\left[\frac{(n + {}^1/_2)^2}{16}\left(\frac{U^{(\mathrm{IV})}(0)}{U''(0)} - \frac{5U'''^2(0)}{3U''^2(0)}\right) + \frac{1}{64}\left(\frac{U^{(\mathrm{IV})}(0)}{U''(0)} - \frac{7U'''^2(0)}{9U''^2(0)}\right)\right]. \tag{19}$$

Additional coefficients in the expansion (17) are computed in similar fashion. Thus,

$$E_3 = \frac{\hbar^3}{\mu^{3/2}\sqrt{U''(0)}}\left\{\frac{\left(n + \frac{1}{2}\right)^3}{288}\left(\frac{U^{(\mathrm{VI})}(0)}{U''(0)} - \frac{17U^{(\mathrm{IV})2}(0)}{8U''^2(0)} - \frac{7U^{(\mathrm{V})}(0)\,U'''(0)}{U''^2(0)} + \frac{75U^{(\mathrm{IV})}(0)\,U'''^2(0)}{4U''^3(0)} - \frac{235U'''^4(0)}{24U''^4(0)}\right) + \frac{5}{1152}\left(n + \frac{1}{2}\right)\left(\frac{U^{(\mathrm{VI})}(0)}{U''(0)} - \frac{67U^{(\mathrm{IV})2}(0)}{40U''^2(0)} - \frac{19U^{(\mathrm{V})}(0)\,U'''(0)}{5U''^2(0)} - \frac{153U^{(\mathrm{IV})}(0)\,U'''^2(0)}{20U''^3(0)} - \frac{77U'''^4(0)}{24U''^4(0)}\right)\right\}. \tag{20}$$

The term E_1 corresponds to a harmonic oscillator, while the subsequent terms are determined by the departure from harmonicity.

Equations (17)-(20) give a better approximation for small n, i.e., for the lower levels.

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Translated by R. T. Beyer
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Magnetic Properties of Cobalt and Manganese Carbonates

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The magnetic properties of $MnCO_3$ and $CoCO_3$ have been investigated in the temperature range 14 to 300° K. It was discovered that below T_c ($T_c = 31.5°$ K for $MnCO_3$; $T_c = 17.5°$ K for $CoCO_3$) the magnetic susceptibility increases abruptly and varies greatly with the field. To explain the observed anomalies, it is postulated that below T_c the carbonates go over to an antiferromagnetic state in which the moments of the sublattices do not fully compensate each other.

BIZETTE[1] discovered antiferromagnetism in siderite containing 78% $FeCO_3$. It seemed of interest to investigate the magnetic properties of other carbonates of elements of the iron group. We have investigated the temperature variation of the magnetic susceptibility of $MnCO_3$ and of an anhydrous sample of $CoCO_3$ in the range 14 to 300°K.

The magnetic properties of manganese carbonate were studied on three specimens. The first specimen used was a commercial sample of "ch.d.a." brand; it was not subjected to drying, because we feared partial dissociation upon drying. The second specimen was the same sample subjected to drying at 160° C, in an ampoule hermetically joined to a vessel containing an absorber ($CaCl_2$). Finally, for the manufacture of the third specimen we used a sample we ourselves had prepared. This sample was obtained by heating for 20 hours at 160° C, in a closed test tube, a mixture of a saturated solution of $MnCl_2$ with $CaCO_3$.[2] The values of susceptibility obtained for the first and third specimens lay systematically lower (by 15 and 18%); this was caused by the presence of sorbed water in the first specimen and of residues of $CaCO_3$ in the third. After introduction of corrections according to weight, the susceptibility values of all three specimens agreed over the whole temperature range within the limits of experimental error.

Two specimens of anhydrous $CoCO_3$ were investigated. We obtained them by the same method as the $MnCO_3$—by heating, in a closed ampoule, a mixture of a saturated solution of $CoCl_2$ with $CaCO_3$. The results obtained for the two samples likewise agreed, within the limits of experimental error, after correction according to weight (13%).

The susceptibility measurements were made by the Faraday method, on apparatus similar to that developed earlier[3]. This apparatus permitted continuous coverage of the wide range of temperatures from 14 to 300° K. The accuracy of the temperature measurement was not worse than ± 0.5° at hydrogen temperatures, and better than ± 0.1° at higher temperatures. The error in the absolute measurement of susceptibility was not greater than 5%; in the relative measurement, less than ± 2%.

The temperature variation of the magnetic susceptibility of both carbonates at high temperatures satisfied the Curie-Weiss law $\chi_m = C_M / (T + \Theta)$. The values of the constants C_M and Θ are given in the table.

TABLE

Compound	Range of validity of Curie-Weiss law	C_M	Θ	μ_{eff}		T_c°K
				Theor.	Exptl.	
$MnCO_3$ . .	75—300°K	4.78	64.5	5.92	6.26	31.5
$CoCO_3$ . . .	50—300°K	1.24	65	3.87	4.34	17.5

In the same Table are given the values of μ_{eff} calculated from our measurements of C_M, and the theoretical values of μ_{eff} calculated on the assumption of complete "quenching" of the orbital moments.[4] The results of the susceptibility measurement below 100° K are given in Fig. 1 for $MnCO_3$ and in Fig. 2 for $CoCO_3$.

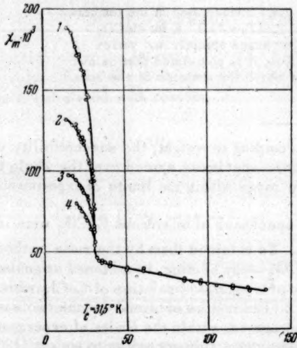

FIG. 1. Temperature dependence of the molar magnetic susceptibility χ_m of manganese carbonate: 1 – H = 490 oersteds; 2 – H = 920 oersteds; 3 – H = 1790 oersteds; 4 – H = 2650 oersteds.

Below some critical temperature T_c, the susceptibility increases abruptly and exhibits a strong dependence on the field. Also observed is a slight hysteresis, which we did not investigate in detail. Figure 3 shows the dependence of the magnetic moment M on the field intensity H for manganese carbonate at temperatures 15 to 40° K. We see that at temperatures below T_c (for fields above about 600 oe), this variation can be represented as the sum of two terms, $M = M_0 + \chi' H$. Similar isotherms were obtained for $CoCO_3$. A field dependence of this type was also observed earlier on a series of other compounds,[5,6] but so far no complete explanation has been found.

From the plotted isotherms $M(H)$, we determined the temperature dependence of M_0 for $MnCO_3$ (Fig. 4) and for $CoCO_3$ (Fig. 5). It has the form characteristic of an ordering curve. For $T \to 0$, however, M_0 approaches a value considerably smaller than the value of magnetic moment to be expected at ferromagnetic saturation ($M_{fer} = N\beta\mu_{eff}$).

For $MnCO_3$: M_0 = 68, M_{fer} = 32000.

For $CoCO_3$: M_0 = 400 to 1000, M_{fer} = 27200.

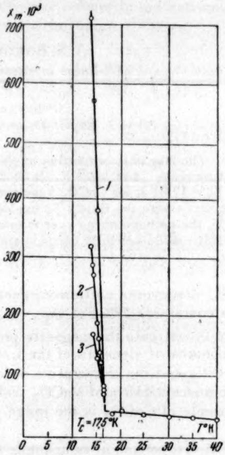

FIG. 2. Temperature dependence of the molar magnetic susceptibility χ_m of cobalt carbonate: 1 – H = 490 oersteds; 2 – H = 1790 oersteds; 3 – H = 2650 oersteds.

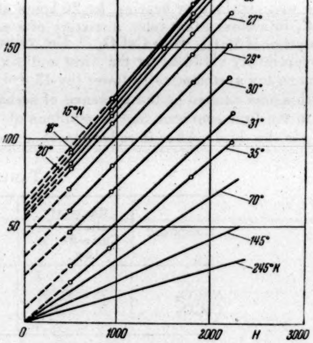

FIG. 3. Dependence of the magnetic moment M on field H at various temperatures, for $MnCO_3$.

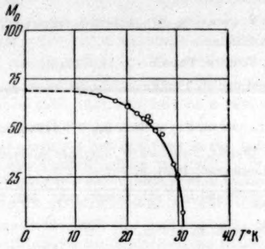

FIG. 4. Temperature dependence of M_0 for $MnCO_3$.

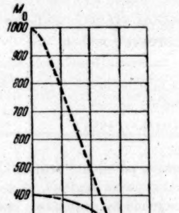

FIG. 5. Temperature dependence of M_0 for $CoCO_3$.

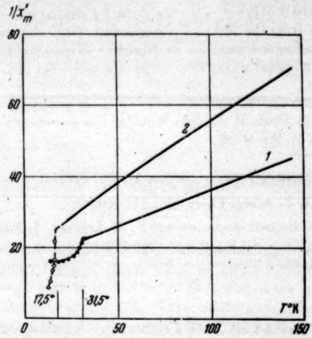

FIG. 6. Temperature dependence of the paramagnetic part of the susceptibility, χ': 1 – $MnCO_3$; 2 – $CoCO_3$.

We do not think that it is possible to explain the observed constant moment M_0, by the presence of ferromagnetic impurities. The agreement of the results obtained on different specimens is evidence against such an explanation.

Also plotted in Fig. 6 are the values of $1/\chi'$ calculated from our experimental data. We see that, in contrast to the usual antiferromagnetics, the susceptibility does not decrease below T_c .

It is possible to explain the results qualitatively by assuming that below T_c , in manganese and cobalt carbonates, an antiferromagnetic ordering develops. In contrast to the usual antiferromagnetics, however, the magnetization vectors of the sublattices in the present case are not exactly antiparallel to each other, but are inclined at a small angle. A similar picture of nonparallel directions of spontaneous magnetization of the sublattices has been established by neutron diffraction for $NiFe_2$.[7] In the body-centered rhombohedral lattice of the carbonates, antiferromagnetic superexchange interaction must occur along the trigonal axis. On this axis the CO_3^{--} groups are distributed between the metallic ions.[8] A small inclination (of the order of a few tens of minutes) of the direction of spontaneous magnetization with respect to the trigonal axis can be explained by the fact that the triangles of the CO_3^{--} groups are turned through 180° in each successive layer. This must lead to the appearance of an uncompensated moment M_0 , directed perpendicular to the trigonal axis. The magnitude of this moment will decrease with temperature along with the magnetization of the sublattices. An alternative basis of explanation of uncompensated moments in antiferromagnetic carbonates might be a difference between the effective magnetic moments of the metallic ions located at the corners and at the center of rhombohedral cell.

Which of these proposals is right might be determined by investigating the magnetic properties of monocrystals. Regrettably, an investigation that we made with a monocrystal of natural $MnCO_3$ (rhodochrosite) showed a quite different behavior of the susceptibility, without noticeable anomalies. Evidently the isomorphic impurities unavoidably present in minerals produce changes of kind in their magnetic properties. In work of Bizette and Tsai,[9] the results of which came to our notice after we had finished our work, a purer mineral—dialogite ($MnCO_3$)—was investigated. Our results agree qualitatively with the data for dialogite. It follows from the data of Bizette and Tsai that an uncompensated moment is directed perpendicular to the trigonal axis; this supports our first proposal.

In closing, the authors convey their profound thanks to Prof. P. G. Strelkov for his constant interest in the work.

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Dispersion Relations for Scattering and Photoproduction

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A derivation is given of the dispersion relations for the following problems: scattering of pions by nucleons (excluding the case of scattering at small but nonzero angles), photoproduction of pions on nucleons, scattering of nucleons and antinucleons by nucleons. The method of consideration is based on the general requirement of the impossibility of the propagation of signals with velocities exceeding that of light, and does not make use of the concept of the *S* matrix for demonstration of the analytical properties of the scattering amplitude. The issue is decided as to whether requirement of microcausality is a necessary condition for the validity of the dispersion relations and it is pointed out that for certain types of violation of causality, the dispersion relations are preserved.

RECENTLY, Goldberger and others[1-4] obtained dispersion relations for the problem of the scattering of mesons from nucleons. In their derivation of the dispersion relations, these authors relied on the concept of microcausality; they made use of a series of general situations of quantum field theory in its present-day formulation. In view of the great generality of the dispersion relations, there is interest in giving another (and, as it appears to us, simpler) derivation of these relations, relying essentially only on the requirement of the impossibility of the propagation of signals with velocities greater than the velocity of light. The method set forth allows us to draw several conclusions on the problem of whether it is necessary, for the existence of dispersion relations, that the propagation velocity of the interaction be smaller than the velocity of light everywhere, even at microscopic distances (of the order of a nuclear distance), or whether it suffices to fulfill this condition only for macroscopic distances. In this case it appears that the dispersion relations are preserved if we assume that the interaction can propagate, not inside the light cone $t^2 - r^2 > 0$, but inside the hyperboloid $t^2 - r^2 > -l_0^2$ (l_0 is a distance of the order of a nuclear distance), i.e., when a condition which violates causality is imposed in the interval.

In the present paper, the dispersion relations are considered for the scattering and photoproduction of pions on nucleons, and for the problem of scattering of nucleons and antinucleons by nucleons.

1. SCATTERING OF PIONS BY NUCLEONS

Let us consider the scattering amplitude (without charge exchange) of $\pi^{\pm}$-mesons by protons, $f_{\pm}(\omega, \theta)$, in which we first limit ourselves to the case of forward scattering*.

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* We shall neglect Coulomb scattering.

Let $f_{\pm}(\omega, 0) \equiv f_{\pm}(\omega)$, given as a function of the frequency ω on the real axis for $\omega > \mu$ (μ = mass of meson, $\hbar = c = 1$), analytically continued over the entire complex plane ω. We shall show that $f_{+}(\omega)$ has no poles in the upper half plane of ω and vanishes sufficiently rapidly on a semicircle of large radius $\Omega \to \infty$ in this half plane.

We shall carry out our analysis in the laboratory system of coordinates, where the nucleon is at rest before and after the collision. In this case, it is appropriate for us to represent the wave function of the nucleon in the form of a wave packet (naturally, of sufficiently large dimensions that the condition of finding the nucleon in a state of rest is satisfied with any degree of accuracy necessary for us). The wave function $\psi_{\pm}(\mathbf{r})$ of the $\pi^{\pm}$-meson that is scattered forward is chosen (at large distances from the scatterer) in the following fashion from the wave function of the incident meson $e^{ikz-i\omega t}$ (origin of the coordinates at the center of the wave packet):

$$\psi_{\pm}(\mathbf{r}) = f_{\pm}(\omega)\, e^{ikz-i\omega t}/r \qquad (1)$$

$$= \int K_{\pm}(t,\ t',\ \mathbf{r},\ \mathbf{r}')\, e^{ikz'-i\omega t'} ds\, dt'.$$

Integration on the right side of (1) is carried out over the region inside the light cone $(t - t')^2 > (\mathbf{r} - \mathbf{r}')^2$, where it suffices to carry out the spatial integration, in accordance with Huygen's principle, over any closed surface surrounding the point $\mathbf{r}$ that does not penetrate the region where the wave function of the pion differs from that function free of the presence of the nucleon. As such a surface we shall choose a plane which is perpendicular to the direction of the momentum of the incident meson (the z axis) and which passes beyond the wave packet [the infinite semicircle closing it does not give a contribution to the integral (1)]. The function K[*] depends on the coordinates of the wave packet, i.e., on the energy of the nucleon in addition to the coordinates $\mathbf{r}$, $\mathbf{r}'$ and the times t, t' of the incident and scattered particles.

For forward scattering in the laboratory system, the initial and final energy of the nucleon is equal to M, the rest mass, and does not depend on ω; therefore, the entire dependence of the right side on ω is determined by the factor $e^{-i\omega t' + ikz'}$. Multiplying (1) by $e^{-ikz + i\omega t}$ and introducing the new variables $\tau = t - t'$, $\boldsymbol{\rho} = \mathbf{r} - \mathbf{r}' = \{\xi, \eta, \zeta\}$, we get (1) in the form

$$\frac{1}{r} f_{\pm}(\omega) = \int\limits_{\tau > \rho} d\tau\, d\xi\, d\eta K_{\pm}(\tau, \rho, \mathbf{r}, t)\, e^{i\omega\tau - ik\xi}. \qquad (2)$$

It is evident from (2) that when ω is found in the upper half plane ($\operatorname{Im}\omega > 0$), an exponentially decaying (with increase in τ) factor appears under the integral of Eq. (2). This factor guarantees the convergence of the integral. It then follows that $f_{+}(\omega)$ cannot have poles in the upper half plane of ω. In order to analyze the behavior of $f_{\pm}(\omega)$ on a large semicircle in the upper half plane of ω, we note that for $\omega \to \infty$, $k \approx \omega - \mu^2/2\omega$ and, consequently, $\operatorname{Im}(\omega\tau - k\zeta) \to \operatorname{Im}\omega(\tau - \zeta) + O(\mu/\omega)\mu\zeta$. Inasmuch as $\tau > \zeta$, then $\operatorname{Im}(\omega\tau - k\zeta) > 0$, and tends toward infinity for $\operatorname{Im}\omega \to \infty$. Thus, $f_{\pm}(\omega)$ disappears, or at least does not increase, on the large semicircle in the upper half plane of ω.

Making use of two properties of $f_{\pm}(\omega)$ in the upper half plane of ω given above (the absence of poles and the vanishing on a semicircle of large radius), it is easy to obtain a correction (with the help of the Cauchy theorem) between the real and imaginary parts of $f_{\pm}(\omega)$:

$$\operatorname{Re} f_{\pm}(\omega) = \frac{1}{\pi} \mathrm{P} \int\limits_{-\infty}^{\infty} \frac{\operatorname{Im} f_{\pm}(\omega')}{\omega' - \omega} d\omega', \qquad (3)$$

where the integration on the right side of (3) in the neighborhood of the pole $\omega' = \omega$ is taken in the sense of the principal value, and all the singularities of the amplitude of $f_{\pm}(\omega)$ which lie on the real axis, are detoured above. Equation (3) is correct if $f_{\pm}(\omega)$ vanishes with sufficient rapidity as $\omega \to \pm\infty$. If this is not so, then the correct result can be obtained by considering the difference $f_{\pm}(\omega) - f_{\pm}(\omega_0)$ in place of $f_{\pm}(\omega)$. Here ω_0 is some fixed frequency. This is equivalent to a subtraction from both sides of (3) of its value for $\omega = \omega_0$.

We now turn our attention to a consideration of the scattering at an angle not equal to zero. In this case, as was shown by Salam[4], it is appropriate to transform to a system of coordinates in which the sum of the initial p and the final p′ of the momenta of the nucleon is equal to zero: $\mathbf{p} + \mathbf{p}' = 0$. It is easy to see that in this system of coordinates, $\omega = \omega'$, $k = k'$, and in place of (1), we will have

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[*] The function K is directly connected with the nucleus by a two-particle equation for the system meson-nucleon. Within the framework of the present formalism, Eq. (1) can be obtained from the properties of the two-particle equation upon consideration of the fact that the nucleon is centered in a small region near the origin of the coordinates.

$$f_\pm(\omega, \theta)/r = \int K_\pm(t, t', \mathbf{r}, \mathbf{r}')\, e^{i\omega(t-t')} e^{-i(kr - \mathbf{kr}')}\, ds\, dt'. \tag{4}$$

The integration in (4) is carried out over a plane which passes through $\mathbf{p}$. Then the function K will depend on ω only through the absolute value of the momentum of the nucleon, $\mathbf{p}$. We denote $q = 2|\mathbf{p}| = 2k \sin(\theta/2)$ and shall consider the scattering amplitude as a function of ω for fixed q. By the same arguments as in the case of forward scattering, we can show that $f_\pm(\omega, q)$ has no poles in the upper half plane of ω. For the investigation of its behavior on the large circle, we denote $t - t' = \tau$, $\mathbf{r} - \mathbf{r}' = \boldsymbol{\rho}$, and make use of the fact that for large $r\,(r \sim \rho \gg r')$ we have $r = \rho + \mathbf{nr}'$. Here $\mathbf{n}$ is a unit vector in the direction of the scattered meson. Then (4) can be written in the form

$$\frac{1}{r} f_\pm(\omega, q) = \int_{\tau > \rho} d\tau\, d\xi\, d\eta\, K(\tau, \boldsymbol{\rho}, \mathbf{r}, t, q) \times \exp\{i[\omega\tau - k\rho] - i(k\mathbf{nr}' - \mathbf{kr}')\}. \tag{5}$$

If ω tends toward infinity and q is fixed, then $\theta \to 0$. In this case we can consider

$$(k\mathbf{nr}' - \mathbf{kr}') \approx k\theta\sqrt{\xi^2 + \eta^2} \approx q\sqrt{\xi^2 + \eta^2},$$

so that the second term in the exponent in (5) is dependent only on q. Inasmuch as K also depends only on q, then the entire dependence on ω for large ω is determined by the factor $e^{i(\omega\tau - k\rho)}$. But $\mathrm{Im}(\omega\tau - k\rho) > 0$ and tends toward infinity for $\mathrm{Im}\,\omega \to \infty$. This permits us to draw the conclusion that $f_\pm(\omega, q)$ vanishes on the large semicircle in the upper half plane of ω. Thus $f_\pm(\omega, q)$, considered as a function of ω for fixed q, must satisfy the relation (3).

Below we shall be interested in the small momenta q transferable to the nucleon, i.e., in small angle scattering. In this case we can consider the nucleon nonrelativistically and describe the scattering amplitude in the form

$$f_\pm(\omega, q) = f_\pm^{(1)}(\omega, q^2) + i\boldsymbol{\sigma}[\mathbf{kk}']\, f_\pm^{(2)}(\omega, q^2), \tag{6}$$

where $\boldsymbol{\sigma}$ is the spin vector of the nucleon. It is clear that the dispersion relations (3) will exist independently for the functions $f^{(1)}$ and $f^{(2)}$. Upon substitution of (6) in (3), the integration over the region $\omega < 0$ reduces to integration over $\omega > 0$ with the help of the relations

$$f_+^{(1)}(-\omega, q^2) = f_-^{(1)*}(\omega, q^2), \qquad f_+^{(2)}(-\omega, q^2) = -f_-^{(2)*}(\omega, q^2). \tag{7}$$

In order to prove (7) we can consider, as proportional to the scattering amplitude, the invariant matrix element $M_{\pm\alpha\beta}(p, p'; k, k')$ which is characterized by a certain arbitrary Feynman diagram (α, β = spin indices). Consideration of the arbitrary diagram (on which we shall not linger) shows that*

$$M_{\pm\alpha\beta}(p', p; -k, -k') = M^*_{\pm\beta\alpha}(p, p'; k, k').$$

Actually $M_\pm$ depends (except for spin factors) only on the three invariants: pp', $(p + p')k$, $(p - p')k$ which, in the system of coordinates for which $\mathbf{p} = -\mathbf{p}'$, are equal, respectively, to

$$pp' = M^2 + {}^1\!/_2 q^2, \quad (p + p')k = \sqrt{M^2 + {}^1\!/_4 q^2}\,\omega, \quad (p - p')k = -{}^1\!/_2 q^2.$$

It is seen that in the given coordinate system the substitution $k \to -k$, $p \to p'$ is equivalent to the substitution $\omega \to -\omega$ for fixed q^2. Then, considering that $[\mathbf{kk}']$, upon the substitution $k \to -k$, $p \to p'$, transforms into itself, we obtain (7). It is evident that for $q^2 = 0$, the first of the relations in (7) transforms into a relation for the forward scattering amplitude in the laboratory system:

$$f_+(-\omega) = f_-^*(\omega). \tag{7'}$$

Substituting (6) and (7) into (3), we get

$$\frac{1}{2}\mathrm{Re}\,[f_+^{(1)}(\omega, q^2) \pm f_-^{(1)}(\omega, q^2)] = \frac{2}{\pi}\int_0^\infty \left\langle {\omega' \atop \omega} \right\rangle d\omega' \frac{{}^1\!/_2\,\mathrm{Im}\,[f_+^{(1)}(\omega', q^2) \pm f_-^{(1)}(\omega', q^2)]}{\omega'^2 - \omega^2}, \tag{8}$$

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* Here it is not necessary to pay any attention to the sign of the infinitely small imaginary contribution $i\epsilon$, inasmuch as the factor which arises from it upon integration over the virtual states is compensated by a corresponding factor which appears in front of the matrix element, and the rule of detouring poles in the integration over ω is determined not by the imaginary contribution but by our requirement on detouring the poles on the upper side.

$$\frac{1}{2}\operatorname{Re}[f_{+}^{(2)}(\omega, q^2) \pm f_{-}^{(2)}(\omega, q^2)] \tag{9}$$

$$= \frac{2}{\pi}\int_0^\infty \left\langle {\omega \atop \omega'} \right\rangle d\omega' \frac{{}^1\!/_2 \operatorname{Im}[f_{+}^{(2)}(\omega', q^2) \pm f_{-}^{(2)}(\omega', q^2)]}{\omega'^2 - \omega^2}.$$

The symbol $< {\omega' \atop \omega} >$ means that for the upper sign (plus) in the square brackets, we must use ω' and for the lower sign (minus), ω. On the right side of (8), (9), there appears an integration over the nonphysical region $0 < \omega' < \omega_{\min} = (\mu^2 + q^2/4)^{1/2}$. In the case of forward scattering ($q^2 = 0$, $\omega_{\min} = \mu$), only the bound states of the meson-nucleon system make a contribution in this region. For computation of this contribution we make use of an expression for the forward scattering amplitude[1]

$$f_{\pm}(\omega) = i\int d^4x e^{ikx} \langle p | T(j_{\pm}^{*}(x), j_{+}(0) | p\rangle. \tag{10}$$

The imaginary part of the amplitude will be

$$\operatorname{Im} f_{\pm}(\omega) \tag{11}$$

$$= \pi\int d\mathbf{x} e^{-i\mathbf{kx}} \sum_n \{\langle p | j_{\pm}^{*}(\mathbf{x}) | n\rangle \langle n | j_{\pm}(0) | p\rangle \times \delta(E_p - E_n + \omega)$$

$$- \langle p | j_{\pm}(0) | n\rangle \langle n | j_{\pm}^{*}(\mathbf{x}) | p\rangle \delta(E_p - E_n - \omega)\},$$

where the summation in (11) is carried out over all the states of the meson-nucleon system. The choice of signs in Eq. (11) is determined by the rule for detouring the poles. We make this choice, keeping in mind the substitution of (11) in (8), where the poles in the integration in ω are detoured on the upper side.

The only bound state in the scattering of pions by protons is the neutron, and the matrix element is (operating in terms of meson theory) the exact vertex part

$$\langle n | j_{-}(0) | p\rangle = ig\sqrt{2}\,\bar{u}(p_n)\,\Gamma_5(p_n, p; k)\, u(p), \tag{12}$$

where g is the renormalized charge. In the given case, $\Gamma_5 = \gamma_5$, inasmuch as all the external momenta correspond to free particles. Further calculations repeat those of Ref. 2 and lead [after subtraction from both parts of (8) of the real part of $f(\mu)$] to the same expressions for the dispersion relations.

In the case of the scattering amplitude at angles not equal to zero, the minimum value of ω, corresponding (for fixed q) to the scattering of a real meson, is $\omega_{\min} = (\mu^2 + q^2/4)^{1/2}$. On the other hand, determining the energy ω_1, beginning with which meson-nucleon states can be realized, from the equality $(M + \mu^2) \leq (k + p)^2$, we find

$$\omega_1 = (M\mu - q^2/4)/(M^2 + q^2/4)^{1/2}.$$

Thus in the interval $\omega_1 < \omega < \omega_{\min}$ on the right side of (8) and (9) there enters the contribution of the nonphysical states of the meson-nucleon system, and this contribution cannot be calculated on the basis of contemporary theory. Therefore, the consideration of the dispersion relations makes sense only for small q^2, where the contribution of the nonphysical states is small.

Let us first consider the spin-flip scattering amplitude $f_{\pm}^{(2)}(\omega, q^2)$, wherein we limit ourselves to the case $q^2 = 0$: $f_{\pm}^{(2)}(\omega, 0) \equiv f_{\pm}^{(2)}(\omega)$. In the nonphysical region $0 < \omega < \mu$, the contribution gives only a bound state of one neutron. Inasmuch as we are only interested in terms proportional to q, we can calculate this contribution, starting with its value for the forward scattering amplitude. Actually, the position of the pole, with accuracy up to terms linear in q, remains the same as in the case of the amplitude of forward scattering (since all the invariants depend only on q^2). The matrix element corresponding to the scattering amplitude can be written in the general case as

$$M(p, p'; k, k') = \bar{u}(p')\{F_1 + k_\mu \gamma_\mu F_2\} u(p), \tag{13}$$

where F_1 and F_2 are functions of the invariants pp', $(p + p')k$, $(p - p')k$, which we can take at the value $q^2 = 0$ in our approximation. Comparing (13) with the value of the matrix element for the case of scattering in the forward direction, we get

$$F_1|_{q^2=0} = 0, \quad F_2|_{q^2=0} = -g^2/M.$$

Computing (13) for these values of F_1 and F_2, we find for the contribution from the bound state, with accuracy up to linear terms in q,

$$\operatorname{Im} f_{+}(\omega, q) \tag{14}$$

$$= (\pi g^2/M)\{\omega - (1/2M)\, i\sigma[\mathbf{kk}']\}\,\delta(\omega - E_p + E_n).$$

The dispersion relations for the scattering amplitude $f_{\pm}^{2}(\omega)$ have the form* [$f = (\mu/2M)g$]

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* Equations (15) coincide with the dispersion relations for the spin-flip scattering amplitude obtained in Ref. 3.

$$\frac{1}{2}\operatorname{Re}[f_{+}^{(2)}(\omega) \pm f_{-}^{(2)}(\omega)] \tag{15}$$

$$= \frac{f^2}{\omega^2 - (\mu^2/2M)^2}\left\langle \begin{matrix} 2\omega/\mu^2 \\ 1/M \end{matrix} \right\rangle$$

$$+ \frac{2}{\pi}\int_{\mu}^{\infty}\left\langle \begin{matrix} \omega \\ \omega' \end{matrix} \right\rangle d\omega' \frac{{}^1/_2 \operatorname{Im}[f_{+}^{(2)}(\omega') \pm f_{-}^{(2)}(\omega')]}{\omega'^2 - \omega^2}.$$

We now go on to the consideration of the nonspin-flip scattering amplitude $f^{(1)}(\omega, q^2)$. As in the earlier case, we shall consider q^2 small, and in the expansion in q^2 we limit ourselves to the zeroth and first terms. The zeroth term evidently gives the dispersion relations for the forward scattering amplitude. For computation of the first term, it is appropriate to expand the region of integration on the right side of (8) to three intervals between the points $\omega = \omega_1$ and $\omega = \omega_{\min}$. For small q^2, the first of these integrals corresponds to the bound state of the system meson + proton , i.e., the neutron, the second to the nonphysical states of the system meson – nucleon and the third to the real states of the system meson – nucleon.

It is easy to see that for small q^2 the integral over the region $\omega_1 < \omega < \omega_{\min}$ is proportional to $q^2 \operatorname{Im} f(\mu, 0)$ and contributes nothing in our approximation [inasmuch as $\operatorname{Im} f(\mu, 0) = 0$]. For the computation of the contribution from the bound state we make use of the general expression for the scattering amplitude[1] and describe it first in the form which it has before the exclusion of the δ-function which describes the law of conservation of momentum:

$$f_{\pm} \sim i\int dx\,dy\,dt\,e^{i\omega t - i\mathbf{kx} + i\mathbf{k'y}} \tag{16}$$

$$\times \langle p' | T[j_{\pm}^{*}(\mathbf{x}, t), j_{\pm}(\mathbf{y}, 0)] | p\rangle$$

$$= \int dx\,dy\,e^{-i\mathbf{kx} + i\mathbf{k'y}} \sum_n \left\{ \frac{\langle p' | j_{\pm}^{*}(\mathbf{x}) | n\rangle\langle n | j_{\pm}(\mathbf{y})\, p\rangle}{E_p - E_n + \omega + i\varepsilon} \right.$$

$$\left. + \frac{\langle p' | j_{\pm}(\mathbf{y}) | n\rangle \langle n | j_{\pm}^{*}(\mathbf{x}) | p\rangle}{E_{p'} - E_n - \omega - i\varepsilon} \right\}$$

(The imaginary contribution was chosen from the condition for the detouring of the poles on the upper side.) Taking the complex conjugate of (16), it is not difficult to see that for these terms in the sum over n where the denominator does not vanish,

$$f_{\pm\,\alpha\beta}^{*(n)}(p, p'; k, k') = f_{\pm\,\beta\alpha}^{(n)}(p', p; k', k),$$

and for the pole terms,

$$f_{\pm\,\alpha\beta}^{*(n)}(p, p'; k, k') = -f_{\pm\,\beta\alpha}(p', p; k', k).$$

Considering the nonspin-flip scattering amplitude ($f_{\pm\alpha\,\beta}^{(1)} \sim \delta_{\alpha\beta}$), and taking it into account that in the coordinate system we have chosen, $f_{\pm}$ depends only on the invariants pp', $(p + p')k$, $(p - p')k$, which do not change upon the substitution $p \to p'$, $k \to k'$, we come to the conclusion that the imaginary part of the amplitude $f^{(1)}$ corresponds to the contribution from those terms in (16) where the denominator vanishes. This permits us to write it in the form

$$\operatorname{Im} f_{\pm}^{(1)}(\omega, q^2) = \pi\int d\mathbf{x}\,e^{-i\mathbf{kx}} \tag{17}$$

$$\times \sum_n [\langle p' (j_{\pm}^{*}(\mathbf{x}) | n\rangle \langle n | j_{\pm}(0) | p\rangle$$

$$\times\, \delta(E_p - E_n + \omega)$$

$$- \langle p' | j_{\pm}(0) | n\rangle \langle n | j_{\pm}^{*}(\mathbf{x}) | p\rangle\, \delta(E_{p'} - E_n - \omega)]|_1.$$

(The subscript 1 denotes that it is necessary to separate out the part which does not contain σ.) With the help of (17), the contribution from the neutron state can be computed in the same way as was done in the case of the forward scattering amplitude. The computations lead to the following expression for the imaginary part of the amplitude [with accuracy up to $(\mu/M)^2$]:

$$\operatorname{Im} f_{+}^{(1)}(\omega, q^2) = -(2\pi g^2/4M^2) \tag{18}$$

$$\times\, [k^2 - {}^1/_2 q^2]\, \delta(\omega - \mu^2/2M - q^2/4M).$$

The expansion of the integral in the region $\omega_{\min} < \omega' < \infty$ in powers of q^2 is elementary. In this case we must replace μ in the lower limit for the same reasons for which the integral over the region $\omega_1 < \omega' < \omega_{\min}$ was omitted. Collecting the results, we obtain, after subtraction from both sides of the equation of its value for $\omega = \mu$, the dispersion relations for $f_{+}^{(1)} \pm f_{-}^{(1)}$:

$$\frac{1}{2}\operatorname{Re}\frac{\partial}{\partial q^2}[f_{+}^{(1)}(\omega, q^2) \pm f_{-}^{(1)}(\omega, q^2)]|_{q^2=0} \tag{19}$$

$$- \frac{1}{2}\left\langle \begin{matrix} \omega \\ \mu \end{matrix} \right\rangle \operatorname{Re}\frac{\partial}{\partial q^2}[f_{+}^{(1)}(\mu, q^2) \pm f_{-}^{(1)}(\mu, q^2)]|_{q^2=0},$$

$$= f^2 \left\langle \begin{matrix} 1/M\mu^2 \\ \omega/\mu^4 \end{matrix} \right\rangle \frac{k^2}{\omega^2 - (\mu^2/2M)^2} + \frac{2k^2}{\pi}\int_{\mu}^{\infty}\left\langle \begin{matrix} \omega' \\ \omega \end{matrix} \right\rangle d\omega'$$

$$\times \frac{{}^1/_2 \operatorname{Im}(\partial/\partial q^2)[f_{+}^{(1)}(\omega', q^2) \pm f_{-}^{(1)}(\omega', q^2)]|_{q^2=0}}{(\omega'^2 - \omega^2)(\omega'^2 - \mu^2)}.$$

The equations (19) can also be obtained from the corresponding equations of Salam[4].

2. PHOTOPRODUCTION OF PIONS ON NUCLEONS

The dispersion relations for the photoproduction of pions on nucleons can be obtained by the method completely analogous to this, with the help of which the relations were found for the scattering of pions on nucleons. We shall consider the amplitudes of photoproduction of π^+-mesons on protons $f_+(\omega)$ and of π^--mesons on neutrons $f_-(\omega)$ at an angle of 0°, as functions of the frequency of the quantum ω. We shall carry out the analysis in a system of coordinates in which the sum of the momenta of the nucleon in the initial and final states is equal to zero: $\mathbf{p} + \mathbf{p}' = 0$. Here, of course, a proper system of coordinates corresponds to each value of the frequency of the quantum ω. Such a choice of a coordinate system brings it about that the amplitude of photoproduction possesses simple properties upon the replacement of ω by $-\omega$. In this system of coordinates the energy of the quantum is equal to the energy of the meson and the initial momentum of the nucleon $\mathbf{p}$ is directed against the momentum of the quantum $\mathbf{k}$ and in absolute magnitude is equal to

$$p = {}^1/_2\,(\omega - \sqrt{\omega^2 - \mu^2}). \tag{20}$$

The threshold of photoproduction lies (in this system) at $\omega = \mu$. Near the threshold, p reaches its maximum value $p_{\max} = \mu/2$, and for large ω ($\omega \gg \mu$), $p \approx \mu^2/2\omega$, i.e., it tends toward zero for $\omega \to \infty$. Inasmuch as, even for $p = p_{\max}$, the kinetic energy of the nucleon $E^{\text{kin}}_{\max} = \mu^2/8M \approx 3$mev is very small, we can regard the nucleons nonrelativistically with very high accuracy.

We note, moreover, one property of the function $p(\omega)$ which follows from its analytic continuation in the region of negative ω:

$$p(-\omega) = -p(\omega) \tag{21}$$

(the points of the loop are detoured above).

The analytical properties of the amplitude of photoproduction can be made clear by writing the expression analogous to (1):

$$f_\pm(\omega)/r = \int\limits_{(t-t')^2 > (\mathbf{r}-\mathbf{r}')^2} K_\pm(t, t'; \mathbf{r}, \mathbf{r}', p) \times e^{i\omega(t-t') - i(qz-hz')}\,dt'ds \tag{22}$$

(q = momentum of the meson), where the integration over ds is carried out in a plane perpendicular to the momentum of the quantum. It follows from (22) that $f_\pm(\omega)$ does not have poles in the upper half plane. Actually, for $\operatorname{Im}\omega > 0$ there appears in the integrand an exponentially decaying factor (for $t' \to -\infty$), which guarantees the convergence of the integral. The function K, by virtue of invariance, can depend on ω only through the combinations $E_{\text{kin}}(t - t') = (p^2/2M)(t - t')$ or pz'. Inasmuch as the integration is not carried out over z', the second of these combinations does not affect the convergence of the integral. So far as the first is concerned, it could change the convergence of the integral only if it were multiplied by a large factor ($\sim M/\mu$) which is completely improbable, becuase E_{kin} must enter on a par with M.

For the investigation of the behavior of $f_\pm(\omega)$ on a large circle in the upper half plane of ω, we must make use of the fact that as $\omega \to \infty$, p tends to zero and $q \to k$. Thus, for sufficiently large ω, the factor $\omega[t - t' - (z - z')]$ appears in the exponent. Its imaginary part is positve for $\operatorname{Im}\omega > 0$ and tends toward infinity for $\operatorname{Im}\omega \to \infty$. Since the dependence of K on ω vanishes for sufficiently large $\omega\,(p \to 0)$, the conclusion as to the vanishing of the function $f_\pm(\omega)$ on the semicircle of large radius in the upper half plane follows directly.

For the photoproduction of mesons at the angle of 0°, the amplitude of photoproduction must have the form

$$f_\pm(\omega) = \boldsymbol{\sigma}\mathbf{e}F_\pm(\omega), \tag{23}$$

where $\mathbf{e}$ is the vector of polarization of the quantum (since the amplitude must be pseudoscalar and must contain $\mathbf{e}$ linearly). The function $F_\pm(\omega)$ has the same analytical properties as the function $f_\pm(\omega)$ and consequently satisfies the relation (3).

The connection of $F(\omega)$ with $F(-\omega)$ can be obtained, as in the case of the scattering of pions, with the help of a consideration of an arbitrary matrix element $M_{\pm\alpha\beta}(p, p'; k, q)$, which corresponds to a certain Feynman diagram. Such a consideration yields the fact that $M^*_{\pm\alpha\beta}(\mathrm{p}', p; -k, -q) = M_{\mp\beta\alpha}(p, p'; k, q)$. M depends only on the invariants pp', $(p + p')k$, $(p - p')k$, which, in our system of coordinates are, respectively, $pp' = E_p^2 + p^2$, $(p + p')k = 2E_p\omega$, $(p - p')k = 2p\omega$. It is evident from these equations that the substitution $p \to p'$, $k \to -k$ is equivalent to the substitution $\omega \to -\omega$. Taking into account Eq. (23) and the

Hermitian character of the matrices σ, we find

$$F^*_+(\omega) = F_-(-\omega). \tag{24}$$

Thus $\frac{1}{2}(F_+ + F_-)$ and $\frac{1}{2}(F_+ - F_-)$ satisfy Eq. (8).

In order to remove from consideration in the integral of the imaginary part of the amplitude the region below the threshold of photoproduction ($\omega < \mu$ in the chosen coordinate system), we can make use of the phase equality pointed out by Fermi[5] for scattering and photoproduction of pions. Inasmuch as for scattering of pions, the phases are purely imaginary for $\omega < \mu$, then for photoproduction in the corresponding states of the nascent pions, the phases will also be purely imaginary for $\omega<\mu$. Consequently, the imaginary parts of the functions $F_\pm$, which are obtained after exclusion (from the expression for the amplitude) of the factor $\boldsymbol{\sigma}\mathbf{e}$, must vanish for $0 < \omega < \mu$.

The contribution of the bound state requires separate consideration. In the case of photoproduction of pions, this could be a proton (or a neutron). The energy E_n of this bound state ought to be determined (in a way similar to what takes place in the scattering of pions by nucleons) by one of the equalities $E_p \pm \omega - E_n = 0$, where the momentum is $\mathbf{p}_n = \mathbf{p}' + \mathbf{k}$ or $\mathbf{p}_n = \mathbf{p} - \mathbf{k}$. With the help of these equalities, it is easy to find that the bound state corresponds to $\omega = 0$, i.e., to the electromagnetic field with constant potential. Such a field, by virtue of gauge invariance, can contribute nothing to the amplitude of photoproduction, so that the contribution from the bound state also vanishes. To sum up, the dispersion relations for the amplitude of photoproduction take the form

$$\frac{1}{2}\operatorname{Re}[F_+(\omega) \pm F_-(\omega)] \tag{25}$$

$$= \frac{2}{\pi}\int_\mu^\infty \left\langle {\omega' \atop \omega} \right\rangle \frac{{}^1\!/_2 \operatorname{Im}[F_+(\omega') \pm F_-(\omega')]}{\omega'^2 - \omega^2}\, d\omega',$$

where the integration on the right begins at the threshold of photoproduction ($\omega = \mu$).

The amplitude of photoproduction of a single pion ought to tend to zero for high energies, as also each amplitude of a definite inelastic process[6,7]. Therefore, the integrals in (25) ought to converge. The dispersion relation for the amplitude of photoproduction of π^0-mesons has a form identical with the relation for $\frac{1}{2}(F_+ + F_-)$.

3. SCATTERING OF NUCLEONS AND ANTINUCLEONS BY NUCLEONS

The dispersion relations for the scattering of nucleons by nucleons were considered in a paper by Fainberg and Fradkin[8]. Their method was analogous to the method of Goldberger[1]. We shall show how these relations can be obtained by means of our method.

We limit ourselves to the case of forward scattering. We shall consider the scattering amplitude of protons by protons f_+ and antiprotons by protons f_-, in the laboratory system of coordinates, as functions of the energy of the incident particle, neglecting the Coulomb interaction. We shall consider both the incident particle and the particle at rest to be polarized in (or against) the direction of the momentum of the incident particle. We introduce the notation: r, s for the polarization of the incident and the second particle before the collision, and r', s' after collision ($r, s, r', s' = \pm 1$).

Proof of the fact that the scattering amplitude has no poles in the upper half plane of ω, and vanishes (or at least does not increase) on a semicircle of large radius, runs identically with that for the case of scattering of pions on nucleons and leads, on the basis of Cauchy's theorem, to a relation similar to (3). In order to investigate the relation of the scattering amplitude of proton-proton for positive ω with the scattering amplitude of antiproton-proton for negative ω, we consider an arbitrary matrix element $M^{\text{nonex}}_{++}(p_1, p_1'; p_2, p_2'; r, r'; s, s')$ (p_1, p_1' are the 4-momenta of the incident particle, p_2, p_2' of the second particle). The nonexchange matrix element can be written in the form

$$M^{\text{nonex}}_{++}(p_1, p_1'; p_2, p_2'; r, r'; s, s')$$
$$= \bar{u}^{r'}(p_1') M_1(p_1', p_1) u^r(p_1)$$
$$\times \bar{u}^{s'}(p_2') M_2(p_2', p_2) u^s(p_2),$$

where, for example, $u^r(p_1)$ is a spinor corresponding to the 4-momentum p_1 and polarization r. We make the substitutions $p_1 \to -p_1$, $p_1' \to -p_1'$, $p_2 \to p_2'$, $s \to s'$ in Eq. (26), and take its complex conjugate. We get

$$M^{*\text{nonex}}_{+}(-p_1, -p_1'; p_2', p_2; r, r'; s'; s) \tag{27}$$
$$= \bar{u}^r(-p_1) M_1(-p_1, -p_1') u^{r'}(-p_1')$$
$$\times \bar{u}^{s'}(p_2') M_2(p_2', p_2) u^s(p_2).$$

The right-hand side of Eq. (27) is the scattering matrix element for the antiproton (with momentum p_1) by the proton (with momentum p_2 and polarization s). Here $u^r(-p_1)$ is the spinor corresponding to the negative energy $-E_1$ and polarization r. The spinor $v(p)$, which represents the wave function of the antiproton, is connected with $u(-p)$ by means of the matrix of charge conjugation $v(p) = u^*(-p)C$. Making use of the explicit form of the matrix C, it is not difficult to prove that in the case of nucleons polarized along (or against) the direction of the momentum, the polarization r of the spinor with negative energy $u(-p)$ corresponds to the polarization $-r$ of the spinor with positive energy $v(p)$. Thus

$$\overset{*}{M}{}^{\text{nonex}}_{++}(-p_1, -p_1'; \ p_2', \ p_2; \ r, \ r'; \ s', s) \qquad (28)$$
$$= M^{\text{nonex}}_{-}(p_1, \ p_1'; \ p_2, \ p_2'; -r_1 - r'; \ s, \ s').$$

For the exchange matrix element there is an analogous relation with just this difference, that the right side corresponds to a diagram in which the nucleon and antinucleon lines exchange their roles.

In the laboratory system of coordinates, $E_2 = E_2' = M$, $\mathbf{p}_2 = \mathbf{p}_2' = 0$ and the vector $\mathbf{p}_1 = \mathbf{p}_1'$ cannot enter linearly since the polarizations are pseudoscalars. Therefore, the substitution $p_1 \to -p_1, p_1' \to -p_1'$, $p_2 \to p_2'$ is equivalent to the substitution $\omega \to -\omega$, which permits us to write for the scattering amplitude [on the basis of Eq. (28)]:

$$f_+(-\omega; r, r'; s, s') \qquad (29)$$
$$= f_-^*(\omega; -r; -r'; s', s).$$

Below we shall be interested only in coherent scattering without change in the spin of each of the particles: $r = r'$, $s = s'$. We denote the amplitude of such scattering by $f_\pm^{r,s}(\omega)$. Then, from (3), with the help of Eq. (29), we get the two relations

$$\frac{1}{2}\operatorname{Re}[f_+^{r,s}(\omega) \pm f_-^{-r,s}(\omega)] \qquad (30)$$
$$-\frac{1}{2}\operatorname{Re}[f_+^{r,s}(M) \pm f_-^{-r,s}(M)]$$
$$= \frac{2}{\pi}(\omega^2 - M^2)$$
$$\times \int^{\infty} \left\langle {\omega' \atop \omega} \right\rangle d\omega' \frac{\tfrac{1}{2}\operatorname{Im}[f_+^{r,s}(\omega') \pm f_-^{-r,s}(\omega')]}{(\omega'^2 - \omega^2)(\omega'^2 - M^2)}.$$

(To assure convergence of the integrals, we subtract from each part of the equation its value for $\omega = M$.)

In order to make clear the properties of the imaginary part of the forward scattering amplitude in the interval $0 < \omega < M$, we return to the general expressions for the amplitudes, which have the form:

$$f_+^{r,s}(\omega) \qquad (31)$$
$$= i\frac{\omega}{2\pi}\int d^4x e^{-ip_1x} \bar{u}_\alpha^r(p_1) \langle p_2, s \,|\, T[\chi_\alpha(x),$$
$$\bar{\chi}_\beta(0)] \,|\, p_2, s\rangle u_\beta^r(p_1);$$

$$f_-^{r,s}(\omega) = i\frac{\omega}{2\pi}\int d^4x e^{-ip_1x} \bar{u}_\alpha^r(-p_1) \qquad (31')$$
$$\times \langle p_2, s \,|\, T[\chi_\alpha(0), \bar{\chi}_\beta(x)] \,|\, p_2, s\rangle u_\beta^r(-p_1).$$

In (31), (31′), $\chi(x)$ denotes the interaction operator standing on the right-hand side of the equation for the nucleon ψ-function in the Heisenberg representation:

$$\gamma_\mu \partial\psi(x)/\partial x_\mu - m\psi(x) = \chi(x).$$

In the case of a pseudoscalar symmetric theory, $\chi(x) = ig\tau_i\gamma_5\psi(x)\varphi_i(x)$. Considering that the interaction is carried out by pseudoscalar mesons, we omit, on the right side of (31), (31′), terms proportional to $\bar{u}^r(p_1)\gamma u^r(p_1)$, which vanish for forward scattering. The imaginary part of the amplitude of scattering can be written in the form of a sum over the entire system $|n>$ of intermediate states:

$$\operatorname{Im} f_+^{r,s}(\omega) = \frac{\omega}{2}\sum_n \int d\mathbf{x} e^{-i\mathbf{p}_1\mathbf{x}} \bar{u}_\alpha^r(p_1) \qquad (32)$$
$$\times [\langle p_2, s \,|\, \chi_\alpha(\mathbf{x}) \,|\, n\rangle \langle n \,|\, \bar{\chi}_\beta(0) \,|\, p_2, s\rangle$$
$$\times \delta(M - E_n + \omega)$$
$$+ \langle p_2, s \,|\, \bar{\chi}_\beta(0) \,|\, n\rangle \langle n \,|\, \chi_\alpha(\mathbf{x}) \,|\, p_2,$$
$$s\rangle \delta(M - E_n - \omega)] u_\beta^r(p_1);$$

$$\operatorname{Im} f_-^{r,s}(\omega) = \frac{\omega}{2}\sum_n \int d\mathbf{x} e^{-i\mathbf{p}_1\mathbf{x}} \bar{u}_\alpha^r(-p_1) \qquad (32')$$
$$\times [\langle p_2, s \,|\, \bar{\chi}_\beta(\mathbf{x}) \,|\, n\rangle \langle n \,|\, \chi_\alpha(0) \,|\, p_2, s\rangle$$
$$\times \delta(M - E_n + \omega) + \langle p_2, s \,|\, \chi_\alpha(0) \,|\, n\rangle \langle n \,|\, \bar{\chi}_\beta(\mathbf{x}) \,|\, p_2,$$
$$s\rangle \delta(M - E_n - \omega)] u_\beta^r(-p_1).$$

[The sign before the second terms in parentheses in Eqs. (32) and (32′) are chosen from the condition of detouring the poles above.] By virtue of the law of conservation of momentum, the total momentum in the intermediate state is $\mathbf{p}_n = \pm \mathbf{p}$. It is not difficult to see that the second term of (32′) is in fact equal to zero. Actually, in this term, only states with two or more nucleons can give a contribution in the sum over n. Since the momentum of such a state must be equal to p, then $E_n > \sqrt{(2M)^2 + p^2}$ and $E_n + \omega$ is always larger than M (even for $\omega < M$). In the first term of Eq. (32), only states with two or more nucleons also can enter into the sum over n, i.e., $E_n > \sqrt{(2M)^2 + p^2}$. It then follows that the quantity $M - E_n + \omega < M - \sqrt{(2M)^2 + p^2} + \omega$ in the δ-function cannot vanish for $\omega < M$. Thus, for $\omega < M$, only the second term of (32) and the first of (32′) can be different from zero.

In the second term of (32), there can enter as intermediate states those of one, two and more mesons. The contribution of the state with a single meson can be calculated precisely, since the matrix element is an exact vertex part*. We have

$$\operatorname{Im} f_{+1}^{r,s} = \frac{\omega}{2} \,|\langle p_1, s \,|\, g\,\overline{\psi}(0)\,\gamma_5\tau_3 u^r(p_1)\,\varphi_3(0)\,|\,n\rangle|^2\,\delta\left(M - \sqrt{\mu^2 + p^2} - \omega\right) \tag{33}$$

$$= -\pi g^2 (\mu/2M)^2 (\omega/M)\,\delta_{r,s}\,\delta\left(\omega - M + \frac{\mu^2}{2M}\right).$$

It is of interest to note that the pole lies only $\mu^2/2M$ (~5 mev in the center-of-mass system) lower than the zero of the kinetic energy. The contribution of intermediate states with two and more mesons cannot be computed on the basis of contemporary theory. It can only be pointed out that these states have especially high energy, so that, for example, the term in the imaginary part, corresponding to a two-meson intermediate state, will differ from zero only for $-(\omega - M) < 4\mu^2/2M$ and, naturally, will not have the character of a δ-function.

The first term in (32′) corresponds to processes of annihilation of the antinucleon with the nucleon. It is not difficult to verify that in the intermediate state here there ought to be at least two pions, so that for $\omega < M$, a term analogous to (33) does not arise.

Besides the pole in the scattering amplitude which corresponds to a single-meson intermediate state, one needs to take into consideration another pole arising from the virtual level (with energy ϵ) of a system of two protons in the ^1S state.

Finally, we get the following dispersion relations for the scattering of protons by protons and antiprotons by protons*:

$$\operatorname{Re}\left[f_{\pm}^{r,s}(\omega) - \frac{1}{2}\left(1 \pm \frac{\omega}{M}\right) f_{\pm}^{r,s}(M) - \frac{1}{2}\left(1 \mp \frac{\omega}{M}\right) f_{\mp}^{-r,s}(M)\right] \tag{34}$$

$$= \mp f^2 \frac{1}{\mu^2} \frac{\omega^2 - M^2}{\omega - M + \mu^2/2M}\,\delta_{r,s} - \frac{2}{\sqrt{M\varepsilon_1}} \frac{\omega - M}{\omega - M + \varepsilon_1} \left\langle {\delta_{-r,s} \atop 0} \right\rangle$$

$$+ \frac{1}{\pi}(\omega^2 - M^2)\left\{\int_0^{M - 4\frac{\mu^2}{2M}} \frac{d\omega'}{\omega'^2 - M^2} \frac{\operatorname{Im} f_{+}^{r,s}(\omega')}{\omega' \mp \omega} + \int_0^M \frac{d\omega'}{\omega'^2 - M^2} \frac{\operatorname{Im} f_{-}^{-r,s}(\omega')}{\omega' \pm \omega}\right\}$$

$$+ \frac{\omega^2 - M^2}{2\pi^2} \int_M^\infty \frac{d\omega'}{\sqrt{\omega'^2 - M^2}} \frac{1}{2}\left\{\frac{\sigma_{+}^{r,s}(\omega')}{\omega' \mp \omega} + \frac{\sigma_{-}^{-r,s}(\omega')}{\omega' \pm \omega}\right\}.$$

As a consequence of the fact that in the first part of the dispersion relations there enter integrals from the imaginary parts in the energy region $\omega < M$, it is not clear whether they can be proved to be any sort of poles. Some interest attaches to the estimate of the first "pole" term in the right-hand side of (34) which arises from the single meson intermediate state. Since this term contains $\delta_{r,s}$ and is proportional to p^2 for small momenta, it must be related to the ^3P state. Estimating the remaining terms in the first part of (34), we can see that for small energies they all (except the third) are considerably smaller than the first (at least in the ratio μ/M). The third term does not contain a small parameter in comparison with the first, but inasmuch as integration over $\omega' - M$ is begun in it with an energy four times larger than the "kinetic" energy of the point of the pole in the first term, and extends to M, a basis is provided for thinking that it also cannot in any appreciable degree compensate the "pole" term at low energies ($\omega_{\mathrm{kin}} \sim \mu^2/2M$). If we consider that the "pole" term cannot compensate the remaining terms at low energies, then with its help we can determine the P-phase in the proton-proton scattering. (It amounts to about 1° at an energy of 5 mev in the laboratory system.) Unfortunately, the experimental data presently available on this problem are sufficiently indeterminate, although they lend support to the idea that the P-phase at these energies is somewhat smaller^9. If making the experimental data more precise shows that the P-phase is actually significantly less than that value which is required by the "pole" terms, then this will mean that the role of the third and fourth terms in (34) (which could not be determined from experiment) is significant even for small energies (except for the trivial case in which all the scattering is determined by a virtual level). This possibly reduces to zero the value of the dispersion relations in the nucleon-nucleon scattering.

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* There can be no intermediate state with a single K-meson because of the conservation of strangeness.

* For the scattering of protons and antiprotons by neutrons, the same relations hold, only with this difference: the pole term from the single meson intermediate state will be twice as large and a contribution is added from the bound state—the deuteron.

4. CONCLUSION

The method carried out above of obtaining the dispersion relations was actually based on only one assumption: the impossibility of the propagation of signals with velocities exceeding that of light, inasmuch as all the other considerations (the replacement of ω by $-\omega$, and the calculation of the contribution from the poles) possess only an auxiliary character and could be replaced by other considerations which do not make use of the concept of the S-matrix in its present-day form. Therefore, the question is of great interest as to whether it is necessary to require the validity of these conditions for microscopic distances or whether it is sufficient to limit oneself to the macroscopic where such an assumption raises no doubt. The conclusion set forth above permits us to bring forth some arguments in support of the following possibility*. Actually, we assume that the region of propagation of the interaction is not limited by the light cone but extends over some small region beyond it, i.e., for example, that the signals can reach a point $\mathbf{r}$, t from all points $\mathbf{r}'$, t' which satisfy the condition $(t-t')^2 - (\mathbf{r}-\mathbf{r}')^2 > -l_0^2$; l_0 is some constant of the order of a nuclear distance. We assume, and this is essential, that the condition which violates causality is superposed in the interval, because in the opposite case it would be difficult to represent that the violation of causality in a small region in one system of coordinates would not become a violation of causality in a large region in another system. Then, for example, in the case of scattering of pions by nucleons, we would have for the amplitude of the forward scattering [in place of Eq. (2)]

$$f_{\pm}(\omega)/r = \int\limits_{\rho^2 - \tau^2 + l_0^2} d\tau\, d\xi\, d\eta\, K_{\pm}(\tau, \rho, \mathbf{r}, t)\, e^{i\omega\tau - ik\xi}. \quad (35)$$

The demonstration of the absence of poles in the upper half plane evidently remains without change. We consider the behavior of $f_{\pm}(\omega)$ on a large semicircle. Let $\xi = \eta = 0$ (this is the worst case). Then it is evident that for large ζ (ζ is a macroscopic distance), the lower limit of the integration over τ is equal to $\tau_{\min} = \zeta - l_0^2/2\zeta$ and consequently, for sufficiently large ζ the correction due to l_0 different from zero can be made arbitrarily small. At first glance, it would seem that, from the presence of a finite mass, it is impossible in Eq. (35) to extend ζ to infinity, since in this case the exponent would be equal to $\exp\{-\frac{1}{2}i[\omega l_0^2/\zeta - \mu^2\zeta/\omega]\}$ (if $\tau = \tau_{\min}$ and $\omega \gg \mu$) and, therefore, for $\omega = i\Omega$ gives an exponentially increasing function. This difficulty could be avoided in the following fashion. We choose a sufficiently large (macroscopic) ζ and let $\omega = \Omega e^{i\varphi}$, $\Omega \sim \mu\zeta/l_0$. Then the exponent will be of the order $\omega l_0 \leq 1$. Along with this, multiplying $f_{\pm}(\omega)$ by a sufficiently rapidly decreasing function, we can become convinced that the integral of $f_{\pm}(\omega)$ over the large semicircle of radius Ω will be very small (in the ratio of some power of l_0/ζ) and the dispersion relations are preserved. In practice, the form of this function, by which it is necessary to multiply $f_{\pm}(\omega)$ is determined by the behavior of $f_{\pm}(\omega)$ on the real axis, i.e., by the behavior of the cross section for large ω. Consequently, in the model considered, the violation of causality in the small region does not change in the course of the proof*.

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* Our attention was called to this point by I. Ia. Pomeranchuk.

* For the case of the scattering of gamma quanta, the conclusion that the dispersion relations do not change if the condition of causality is violated in the small, can be obtained also from the results of Ref. 10 if, in obtaining Eq. (3.19) of this paper from (3.18), we introduce a condition imposed in the interval that violates causality.

However, we can choose the condition that violates causality in some fashion which does not impose it in the interval. Here, generally speaking, one chould expect that the dispersion relations will no longer be maintained.

We have therefore come to the following conclusion: if the experimental data are in contradiction with the dispersion relations, then this will mean that at small distances, the propagation of signals with velocities exceeding that of light can go on; at the same time, in accord with experimental data with dispersion relations, we cannot exclude the violation of causality at small distances, in particular the propagation of the reaction between two points lying not inside the light cone but inside the hyperboloid appears to be possible.

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On a Singularity of the Field of an Electromagnetic Wave Propagated in an Inhomogeneous Plasma

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The effect of field growth is investigated in a region where the plasma dielectric constant becomes zero. The problem of the absorption influence is fully explained. The relationship between this effect and plasma resonance is established.

IN solving the problem of propagation of electromagnetic waves in an inhomogeneous plano-stratified medium the simplest case is that of normal incidence. Under the conditions of complete reflection it is most convenient to use the linear approximation of the dielectric constant $\epsilon(z)$ in the neighborhood of its zero (point of reflection). In fact, the consideration of this simplest case enables one to explain completely the field of a standing wave in the reflection region (see Ref. 1, Sec. 66). An analogous situation occurs for oblique incidence.

Zhekulin[2] carried out a detailed investigation of solutions describing the oblique incidence of radio waves on a plano-stratified isotropic ionosphere. In such a medium waves with different polarizations of the electric vector E (perpendicular and parallel to the incidence plane) are propagated independently of each other. It turns out that the reflection problem of the wave, with an electric vector perpendicular to the incidence plane, does not differ in principle from the well known case of normal incidence. They differ only in the displacement of the incident wave reflection level. However, the equation describing the wave with a different polarization of the electric vector is of a more specific type; in this case, the point at which the dielectric constant of the medium ϵ, $\epsilon(z)$ becomes zero is a singularity. Zhekulin showed that the condition for the solution to become zero at infinity (in the region of negative values for ϵ) is not compatible with the requirement that the field be finite at the zero of ϵ. In trying to remove this solution singularity, the author replaces (without any basis) the function $\epsilon(z)$, which changes sign at a definite point, by a positive function which does not become zero anywhere.

Forsterling and Wuster[3,4] discussed in greater detail the singularities of the field for oblique incidence of the wave. From the analysis of approximate solutions, valid in a relatively small neighborhood of the zero of the function ϵ, it was established that the field component E_z becomes infinite as $1/\epsilon$ at this point, while the component E_y has a logarithmic singularity. The abrupt growth of field strength in the region where ϵ takes on infinitesimally small values leads to the conclusion that the field description with the aid of the usual dielectric constant becomes impossible. This is clear from the fact that the motion of the electrons cannot be harmonic under the influence of a field with an abrupt space innomogeneity. This case can be accounted for by the fact that the equations describing the field in the medium become nonlinear, and, during the propagation of a wave with a definite frequency, there waves arise in such a medium of other frequencies (higher harmonics).[4]

However, in the above mentioned papers, there remained unexplained the problems of the amplitude in a growing field within an absorbing medium, and the physical nature of this singularity in a medium without absorption.

This paper is devoted to a detailed discussion of these problems.

1. SINGULARITIES OF THE ELECTROMAGNETIC WAVE FIELD WITH OBLIQUE INCIDENCE ON A LINEAR LAYER

Let us consider the problem of the electromagnetic wave incidence on an inhomogeneous medium, the properties of which are a function of a single coordinate z only. We shall assume that the normal to the wave front of the plane wave lies in the y,z plane. In such a case the equation of interest to us, describing the wave with its components H_x, E_y E_z, is written in the form (see, for instance, Refs. 1,2)

$$\frac{d^2w}{dz^2} - \frac{1}{\varepsilon'(z)}\frac{d\varepsilon'(z)}{dz}\frac{dw}{dz} + k_0^2(\varepsilon - q^2)w = 0. \quad (1)$$

The function $w(z)$ is related to the H_x component by the relationship

$$H_x = w(z)\, e^{i(\omega t \div k_0 q y)}. \quad (2)$$

The remaining field components are determined from the equations

$$E_y = \frac{1}{i' k_0 \varepsilon'(z)}\frac{\partial H_x}{\partial z}, \quad E_z = -\frac{1}{i k_0 \varepsilon'(z)}\frac{\partial H_x}{\partial y}. \quad (3)$$

In Eq. (1) and in formulas (2), (3) there are also found the following symbols: $k_0 = \omega/c$ is the wave number in vacuum; $q = \sin\theta_0$, where θ_0 is the value of the incidence angle at the boundary of the inhomogeneous layer. The dielectric constant of the ionized gas $\epsilon^1(z)$ is expressed by the formula (see Ref. 1, Sec. 57)

$$\varepsilon'(z) = 1 - 4\pi e^2 N(z) / m\omega^2 (1 - i\nu_{\mathrm{eff}}/\omega),$$

where e,m is the charge and mass of an electron, N is electron density, ν_{eff} is effective number of electron collisions with neutral molecules. Assuming a small absorption, that is $\nu_{\mathrm{eff}}/\omega \ll 1$, one can write the approximation

$$\varepsilon'(z) = \varepsilon(z) - (i\omega_0^2/\omega^2)\nu_{\mathrm{eff}}/\omega.$$

Here $\omega_0^2 = 4\pi e^2 N/m$ is the natural frequency of the plasma oscillations. Furthermore, by assuming that the absorption varies little with altitude, one can take the imagniary part of $\epsilon^1(z)$ as equal to its constant value at the point where $\omega = \omega_0$. In such a case

$$\varepsilon'(z) = \varepsilon(z) - i\nu_{\mathrm{eff}}/\omega,$$

and for the linear dependence $\epsilon(z) = 1 - \omega_0^2$

$$\varepsilon^*(z) = -\alpha z - i\nu, \qquad \nu = \nu_{\mathrm{eff}}/\omega. \quad (4)$$

The origin of coordinates is chosen so that with $z > 0$ $\epsilon(z) < 0$, and with $z < 0$ $\epsilon(z) > 0$.

The differential equation (1) is now written in the form

$$\frac{d^2w}{dz^2} - \frac{1}{x}\frac{dw}{dz} - \rho^2(x + q^2)w = 0, \quad (5)$$

where $x = \alpha z + i\nu$, and $\rho = k_0/\alpha$. It is easy to observe that the form of Eq. (5) does not change when going through $\nu = 0$.

The only difference for the problem in which absorption is taken into account is that here the "mathematical" point of reflection $x = -q^2$ corresponds to complex values of the z-coordinate: in the complex plane x, the real values of z

lie on a straight line passing in the upper half-plane parallel to the real axis at a distance equal to ν.

Furthermore, let us note that for a medium with slowly changing properties the parameter of Eq.(5) $\rho >> 1$. Thus, in the F-layer of the ionosphere $(\alpha \sim 10^{-7})\rho \sim 3 \times 10^4$ for the frequencies $\omega \sim 10^8$.

In Refs. 3,4 it is shown that the solution of Eq. (5), satisfying the necessary physical requirements, takes on some value not equal to zero at a point where (ϵ^1 / z) becomes zero $(x = 0)$. Therefore, in accordance with (3), the vertical component of the electric field

$$E_z = -(qw / \epsilon')\, e^{i(\omega t + h_0 q y)} \tag{6}$$

becomes infinite at this point. The nature of this singularity depends on the behavior of $\epsilon^1 (x)$. Thus, for a linear layer, E_z becomes infinite as $1/x$ and the E_y component has a logarithmic singularity.[4]

This singularity is located on the real axis only for $\nu = 0$. In taking into account absorption the maximum value of E_z will be equal to

$$|E_z|_{z=0} = q\,|w(0)|/\nu$$

and can be very large for sufficiently small values of ν. With this, the magnitude of the field depends to a considerable extent on the values taken on by the function ω (0). This function depends on the angle of incidence and thus determines the magnitude $(Ez)_{z=0}$ in the entire interval of values for the parameter $q = \sin \theta_0$.

First, let us attempt to determine the form of this function for large angles of incidence when the reflection point $x = -q^2$ and the particular point $x = 0$ are separated by a considerable distance. For the sake of convenience we shall investigate [instead of Eq. (5)] the equation

$$(d^2u / dx^2) - [\rho^2 (x + q^2) + 3/4x^2]\, u = 0, \tag{7}$$

which is satisfied by the function

$$u(x) = w(x) / \sqrt{x}. \tag{8}$$

We shall assume that the distance between the reflection point $x = -q^2$ and the special point $x=0$ of the Eq. (7) is much larger than a wavelength. For a medium with slowly changing properties $(\rho >> 1)$ this takes place for values of q^2 which are not too small. Then the approximate solution of Eq. (7) (valid everywhere except for a small region around the point $x=0$ and representing a standing wave to the left of $x=-q^2$) can be written in the form[5]:

$$u = \sqrt{\pi\rho/2}\; e^{i\pi/2} \sqrt{s/s'}\, H^{(1)}_{1/3} (is), \tag{9}$$

$$s = \rho \int_{-q^2}^{x} \sqrt{x + q^2}\, dx \tag{10}$$

$$= \frac{2}{3}\rho (x + q^2)^{3/2}; \quad s' = ds/dx,$$

where $H_{1/3}$ is Hankel's function of order 1/3. The constant which appears in the solution of Eq. (9) is chosen so that the amplitude of the incident wave field at the boundary of the inhomogenesous layer $[\epsilon^1 (z) = 1]$ is equal to unity.

Another approximate solution, valid to the right of the reflection point, can be obtained by using the method proposed in Ref. 5. Let us introduce a new independent variable

$$\xi = \rho \int_0^x \sqrt{x + q^2}\, dx = {}^2/_3\, \rho\, [(x + q^2)^{3/2} - q^3].$$

It is easily shown that the function

$$u^* = B \sqrt{\frac{\xi}{d\xi/dx}}\, H^{(1)}_{1} (i\xi) \tag{11}$$

satisfies the equation

$$\frac{d^2u^*}{dx^2} - \Big[\rho^2 (x + q^2) + \frac{3}{4}\left(\frac{1}{\xi}\,\frac{d\xi}{dx}\right)^2 - \frac{5}{16(x + q^2)^2}\Big]\, u^* = 0. \tag{12}$$

For small $x\xi = \rho q x$ Eq.(12) also has the same singularity at the point $x=0$ as the fundamental Eq. (7). Besides, for large values of the parameter ρ, Eqs. (12) and (7) differ very little from each other, if one excludes from consideration some region around the point $x=-q^2$ where the function $(x + q^2)^{-2}$ begins to grow abruptly. Consequently, the solutions of these equations in the region $x > -q^2$ will also differ little from each other. At the same time, Eq. (11) will approximate the solution which approaches zero as $x \to +\infty$ [in the region of negative values of $\epsilon(z)$]. We note that the solution which was obtained and investigated in Ref. 4 can be obtained from Eq. (11) by assuming that for small x, $\xi = \rho q x$.

We obtained the approximate solutions (9) and (10) giving the asymptotic behavior of the desired solution (for $\rho \to \infty$) for the various regions of the variable x values: to the left of $x = 0$ [Eq. (9)] and to the right of $x = -q^2$ [Eq. (11)]. In the interval $= -q^2 < x < 0$ both approximations are correct, which enables one to combine these solutions so that they would yield the same particular solution of our problem. The necessary calculations are materially simplified if one uses the fact that both approximate expressions have the same asymptotic behavior in the above mentioned interval. Actually, using the asymptotic representations for the functions $H_{1/3}^{(1)}(is)$ in (9) and $H_1^{(1)}(i\xi)$ in Eq. (11) we obtain for $s \gg 1$ to the right of the point of $x = -q^2$

$$u \approx -\sqrt{\rho/s'}\,e^{-s+i\pi/4} \tag{13}$$

and to the left of $x-0$ for $|\xi| \gg 1$,

$$u^* \approx -B\sqrt{2/\pi\zeta'}\,e^{|\xi|}, \tag{14}$$

expressions accurately agreeing with each other up to the constant multiplier, because $ds' = d\xi'$ and

$$s = \rho\int_{-q^2}^{x}\sqrt{x+q^2}\,dx = s_0 - |\xi|; \quad s_0 = \frac{2}{3}\rho q^3 \tag{15}$$

in the interval $-q^2 < x < 0$. Comparing (13) and (14), we find the value of the constant B

$$B = \sqrt{\pi\rho/2}\,e^{i\pi/4 - s_0}. \tag{16}$$

Thus on the basis of (11) and (16) we can now write the final formula, which gives the behavior of function $w(x)$ in the region $x > -q^2$, in the following form:

$$w(x) = \sqrt{x}u^* = \sqrt{\pi\rho/2}\cdot\sqrt{x\zeta/\zeta'}\,e^{i\pi/4 - s_0}H_1^{(1)}(i\xi). \tag{17}$$

The remaining components of the field can be calculated from formulas (2) and (3).

Here we shall be interested primarily in the behavior of the E_z component in the vicinity of the $\epsilon'(z)$ zero. Formula (17) shows that $w(x)$ converges to a finite value for $x \to 0$:

$$w(0) = \sqrt{2/\pi\rho}\,e^{-s_0 - i3\pi/4}/\cdot q \tag{18}$$

and consequently, with $|x| \ll 1$, the behavior of $|E_z|$ according to (6) is given by the formula

$$|E_z| = \sqrt{2/\pi\rho}\,e^{-s_0}/|x| \tag{19}$$

with a maximum value (in a medium with an absorption $x = \alpha z + i\nu$)

$$|E_z|_{z=0} = \sqrt{2/\pi\rho}\,e^{-s_0}/\nu. \tag{20}$$

It is to be remembered that the final formulas are applicable only for large angles of incidence; Eq. (18) for $q \to 0$ gives an obviously incorrect result. However, for the upper ionosphere layers ($\rho \gg 1$) the approximation formulas are applicable up to angles of incidence θ_0 of the order 4–5°, and, as it can be easily verified, under these conditions, the effect of field growth near the point $z = 0$ will be negligible ($s_0 \gg 1$). The field will take on large values only for unrealistically small ν_{eff}. At the same time, the presence of the singularity at the point where $\epsilon = 0$ does not affect the behavior of the field in the region below the reflection point, that is, the reflection of the wave having an E_z component, and under these conditions of oblique incidence takes place in the same manner as in the case of the wave with an electric vector perpendicular to the plane of incidence.

Furthermore, let us note that in the case of normal incidence ($q = 0$) $E_z = 0$. Consequently, for some small angle of incidence the growth of the field will be a maximum. In connection with this, the behavior of the function $w(0, q)$ for all incidence angles is of interest. Earlier we determined the form of this function for values of q that were not too small [Eq. (18)]. Besides, it is easy to obtain the value of this function for $q = 0$. Actully, the solution of equation (5) converging to zero for $x \to +\infty$ can be written in the form[6]

$$w(x, 0) = 2\sqrt{\rho/3\pi}\,xK_{2/3}\left({}^2/_3\,\rho x^{3/2}\right), \tag{21}$$

where $K_{2/3}$ is MacDonald's function. The constant multiplier in the solution of (21) is chosen so that it includes the incident wave with an amplitude equal to unity at the boundary of the inhomogeneous layer. Using the derivative from the Airy function and its relationship with Bessel's functions[7] let us write the solution of (21) in another form.

$$w(x,\ 0) = -2\rho^{-1/3}v'(t); \quad (t = \rho^{2/3}x).$$

Consequently, the value of the function $w\ (0,q)$ for $q = 0$ (normal incidence) will be equal

$$w(0,\ 0) = -2\rho^{-1/3}v'(0). \qquad (22)$$

Let us look for the approximate expression of this function in the entire range of values of the parameter q. First we shall find the approximate expression $w\ (x,q)$ for small values of x.

In order to do this we shall make explicit the behavior of the function u satisfying Eq. (7). If one looks for its solution in the form

$$u = u_1(s(x))\,u_2(x),\ s(x) = {}^2/_3\,\rho\,(x + q^2)^{3/2}, \qquad (23)$$

then for u_1 and u_2 we obtain the equation

$$\frac{d^2u_2}{dx^2} + 2\frac{du_2}{dx}\frac{ds}{dx}\frac{u_1}{u_1} \qquad (24)$$

$$+\left[\frac{u_1''}{u_1}\left(\frac{ds}{dx}\right)^2 + \frac{u_1'}{u_1}\left(\frac{d^2s}{dx^2}\right)\right.$$

$$\left. - \rho^2(x + q^2) - \frac{3}{4x^2}\right]u_2 = 0,$$

in which we require that

$$\frac{u_1''}{u_1}\left(\frac{ds}{dx}\right)^2 + \frac{u_1'}{u_1}\frac{d^2s}{dx^2} - \rho^2(x + q^2) = 0$$

(here the primes mean differentiation with respect to s). Using Eq. (23), we shall express the last in equation in the form

$$\frac{d^2u_1}{ds^2} + \frac{1}{3s}\frac{du_1}{ds} - u_1 = 0.$$

The required particular solution converging to zero for $s \to +\infty$ is

$$u_1 = s^{1/3}K_{1/3}(s). \qquad (25)$$

Let us substitute this expression in Eq. (24). Then

$$\frac{d^2u_2}{dx^2} + 2f(x)\frac{du_2}{dx} - \frac{3}{4x^2}u_2 = 0, \qquad (26)$$

where $f(x)$ denotes the function

$$f(x) = (ds\,/\,dx)\,u_1'\,/\,u_1$$

$$= -\rho\,(x + q^2)^{1/2}K_{2/3}(s)\,/\,K_{1/3}(s).$$

Limiting the investigation of the solution to a small region around the point $x = 0$, and using the fact that $f(x)$ when compared to the solution itself is a slowly varying function we can consider it as a constant and equal to the value for $x = 0$. Then, denoting the positive constant by

$$-f(0) = q\rho K_{2/3}(s_0)\,/\,K_{1/3}(s_0) = b; \qquad (27)$$

$$(s_0 = {}^2/_3\,\rho q^3),$$

we obtain the following approximate equation for $u_2\ (x)$:

$$d^2u_2\,/\,dx^2 - 2b\,du_2\,/\,dx - \frac{3}{4x^2}u_2 = 0.$$

Its solution can be written by means of known functions.[6] If one chooses the particular solution

$$u_2 = \sqrt{xb}\,K_1(bx)\,e^{bx} \qquad (28)$$

(K_1 is MacDonald's function of the first order), which for $bx << 1$ approaches a constant value, then the final desired function $w\ (x_2\ q)$ is written thus:

$$w(x,\ q) = \sqrt{x}u \qquad (29)$$

$$= As^{1/3}K_{1/3}(s)\sqrt{bx^2}e^{bx}K_1(bx),$$

where the arbitrary constant A must be chosen so that approximation (29) would agree in the small region around the point $x = 0$ with the particular solution in which the amplitude of the incident wave is equal to unity at the boundary of the inhomogeneous layer. If $s_0 >> 1$, then from (29) for $w\ (0,\ q\ )$ we get (18) accurate up to the constant.

A simple calculation gives

$$A = 2\,({}^2/_3\,\rho)^{1/6}/\pi \qquad (30)$$

and finally for the function $w\ (0,q)$

$$w(0,\ q) = (2/\pi)\,({}^2/_3\,\rho)^{1/6}\,s_0^{1/3}K_{1/3}(s_0) \qquad (31)$$

$$\times\{K_{1/3}(s_0)\,/\,[q\rho K_{2/3}(s_0)]\}^{1/2}.$$

If the Airy function is used,

$$v(t)=\sqrt{t/3\pi}K_{1/3}\left(\frac{2}{3}t^{3/2}\right); \qquad (31)$$

$$v'(t)=-t(3\pi)^{-1/2}K_{2/3}({}^{2}/_{3}t^{3/2})$$

(see, for instance, Ref. 7) then $w(q)$ can be written in the following form which is convenient for computations:

$$w(q)=2\sqrt{2/\pi}\,\rho^{-1/6}v(t)\sqrt{v(t)/-v'(t)}; \qquad (32)$$

$$t=q^2\rho^{2/3}.$$

From the calculations it can be seen first that function (32) must be a good approximation to the true value of $w(q)$ for angles of incidence which are not close to zero. A relatively large error can be expected for $q\to 0$. However, a comparison of this function with its exact expression for $q=0$ (22) shows that formula (32) gives a relatively good description of the true behavior of $w(q)$ even for very small incidence angles. In fact, the ratio of the limiting values is given by

$$w(0)_{\text{exact}}/w(0)_{\text{approx}}$$

$$=\sqrt{\pi/2}\,(-v'(0)/v(0))^{3/2}\approx 0.8,$$

and, consequently, one can be sure that the error from the approximate expression (32) does not exceed 20% in the interval of q values from zero to some small magnitude beyond which the error of this approximation is negligible.

With Eq. (32) we can now describe the behavior of the field component E_z of interest to us in the neighborhood of the point where the dielectric constant of the medium become zero ($x=0$). In accordance with (6), for small values of (x),

$$|E_z|=\frac{|qw(q)|}{|x|},$$

$$|qw(q)|=2\sqrt{2/\pi\rho}\sqrt{t}\,v(t)\sqrt{v(t)/-v'(t)},$$

($t=\rho^{2/3}q^2$) or, introducing a new parameter $\tau=\sqrt{t}=\rho^{1/3}q$,

$$|qw(q)|=\frac{4\tau v(\tau^2)}{\sqrt{2\pi\rho}}\sqrt{\frac{v(\tau^2)}{-v'(\tau^2)}}=\frac{\Phi(\tau)}{\sqrt{2\pi\rho}}\cdot \qquad (33)$$

Consequently

$$|E_z|=\Phi(\tau)/\sqrt{2\pi\rho}\,|x|; \qquad (x=\alpha z+i\nu) \quad (34)$$

and takes on a maximum value for $z=0$ equal to

$$|E_z|_{z=0}=\Phi(\tau)/\sqrt{2\pi\rho\nu}. \qquad (35)$$

The dependence of the maximum magnitude $|E_z|$ on the angle of incidence is thus determined by the function $\Phi(\tau)$, the graph of which is shown in Fig. 1; here we also introduce (parallel to the parameter τ axis) a scale in degrees for the incidence angle θ_0 for $\alpha=10^{-7}$ and $\omega=2\pi\times10^7$. It is essential to note that $\Phi(\tau)$ takes on a value of the order of unity for a very narrow interval of the incidence angle values, with the curve maximum equal to 1.2 corresponding to $\theta_0=1.5°$, and when $\theta_0=5°$, $\Phi(\tau)\sim10^{-4}$.

Let us evaluate, by means of Eq. (35), the values reached by the field $|E_z|$ in an isotropic plasma.[1] For the data of the E layer in the ionosphere one can take $\alpha=10^{-6}$, $\omega=6\pi\times10^6$ ($\lambda_0=100$ m). The maximum magnitude $|E_z|$ for $\nu_{\text{eff}}=10^5$ will be $|E_z|_{z=0}\approx3$, and for $\nu_{\text{eff}}=10^4$: $|E_z|_{z=0}\approx30$.

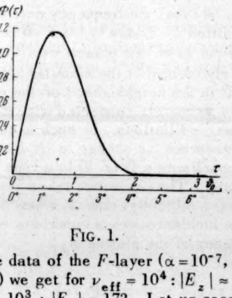

FIG. 1.

For the data of the F-layer ($\alpha=10^{-7}$, $\omega=2\pi\times10^7$, $\lambda_0=30$m) we get for $\nu_{\text{eff}}=10^4$: $|E_z|\approx17.3$, and for $\nu_{\text{eff}}=10^3$: $|E_z|\approx173$. Let us recall that the boundary of the inhomogeneous layer $|E|=1$ and $|E_z|=|E|\sin\theta_0=q$.

Summarizing the investigation, it can be said that for an isotopic plasma with slowly changing properties the growth effect of the field in the region of small $\epsilon(z)$ values plays no role in the case of large incidence angles, but becomes appreciable for small angles $\theta_0\sim2$–$3°$ ($\nu_{\text{eff}}\leq10^3$. In the latter case the presence of the point at which $\epsilon=0$ materially changes the solution form beyond the reflection point, and the growth of the field intensity of the standing wave is not smoothed out by the existing absorption.

2. THE APPROXIMATE CALCULATION OF THE EFFECT OF PLASMA WAVES

We have limited ourselves heretofore to a discussion of collision effects whose calculation leads naturally to a finite value for the field of the electromagnetic wave at the point where $\epsilon = 0$. But in a medium with a small absorption, there remains the anomalous behavior of corresponding solutions and the unexplained problem of the true behavior of the field, since in an inhomogeneous plasma an important role can be played by other factors; their calculations, like the calculation of collisions, will lead to the removal of the singularity discussed previously. The possibility of the existence of other factors (besides collisions and nonlinear phenomena) leading to a finite value of the field intensity at the zero of $\epsilon(z)$ becomes apparent from the following considerations.

The characteristic behavior of the vertical field component E_z of the electromagnetic wave propagated in a plano-stratified ionized medium suggests the idea of relating this phenomenon with definite resonance characteristics inherent in a quasineutral plasma. It is well known (Ref. 1, Sec. 63) that the frequency determined from the condition $\epsilon = 1 - 4\pi e^2 N/m\omega^2 = 0$ is the frequency of the so-called plasma oscillations. It is completely clear that the singular behavior of the field E in the neighborhood of the point where $\epsilon = 0$ is intimately connected with the excitation of these oscillations. In such a case, the function representing the change in $|E_z|^2$ as dependent on the coordinate x (Fig. 2) is a type of a resonance curve. The resonance takes place at the point $x = 0$ [$\epsilon(0) = 0$], that is, where the frequency of the incident wave ω coincides with the natural frequency of the plasma ω_0.

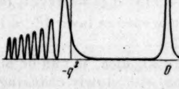

FIG. 2.

Of course, such a dependence of $|E_z|^2$ on a coordinate is characteristic for an idealized problem in which we neglect all sorts of energy dissipation in the standing electromagnetic wave. Naturally, the calculation of collisions results in the elimination of the infinite values for the field. At the same time the behavior of the vertical component in the vicinity of the resonance point is given by the function [see Eq. (34)].

$$|E_z|^2 = A\,[(\alpha z)^2 + (\nu_{\text{eff}}/\omega)^2]^{-1}; \quad A = \text{const}.$$

The width of this "resonance curve", as in the oscillation circuit, is determined by absorption and is equal to $z' = \nu_{\text{eff}}/\omega\alpha$.

However, in an inhomogeneous plasma, another energy dissipation mechanism is possible, namely, the generation of plasma waves. The presence of an abruptly changing longitudinal component of the electric field results in the appearance of a space inhomogeneity in the electron gas. In each section of the medium excited in this manner, the electrons experience oscillations (under the influence of the wave's electric field), the amplitude of which increases with the approach to the resonance point ($x = 0$). Actually, these local oscillations are not independent: every change in electron density in one of the medium's regions is transmitted to its neighbor through electronic pressure; the calculation of the latter results in the appearance of plasma waves which carry away some fraction of the standing electromagnetic wave energy. Finally, the energy associated with the plasma wave is used up in the heating of the gas.

In the more general formulation of the problem we must take into account the possible emergence of plasma waves which, undoubtedly, will result in the elimination of the singularity in the solution and in the finite field value to the resonance point. The following approximate calculations explain the effect of plasma waves on the behavior of the electric field's vertical component in the vicinity of this point.

For the oblique incidence of an electromagnetic wave on a plano-layered isotopic medium the properties of which depend on the coordinate z, the field equations can be written in the form

$$\partial H_z/\partial y - \partial H_y/\partial z = ik_0(E_x + 4\pi P_x), \qquad (36a)$$

$$\partial E_x/\partial z = -ik_0 H_y; \quad \partial E_x/\partial y = ik_0 H_z;$$

$$\partial E_z/\partial y - \partial E_y/\partial z = -ik_0 H_x, \qquad (36b)$$

$$\partial H_x/\partial z = ik_0(E_y + 4\pi P_y),$$

$$\partial H_x/\partial y = -ik_0(E_z + 4\pi P_z);$$

$$\operatorname{div}(\mathbf{E} + 4\pi\mathbf{P}) = 0 \qquad (36c)$$

(it is assmed that the field is a function of time according to the law $e^{i\omega t}$ and is independent of the x-coordinate). To the system (36) one must also add the relationship between the electric field strength E and the polarization P. This relation can be easily obtained for the problem at hand by using the equation for the motion of electrons under the influence of the field E and electron pressure

$$mN\ddot{\mathbf{r}} = -\varkappa T\nabla N + eN\mathbf{E} \tag{37}$$

($\varkappa$ is Boltzmann's Constant, T is absolute temperature, N is electron concentration) and the equation of continuity

$$(\partial N/\partial t) + \operatorname{div} N\dot{\mathbf{r}} = 0. \tag{38}$$

Equations (37) and (38) are for processes harmonic in time. By taking into account the fact that

$$\partial N/\partial t \approx \partial n/\partial t = i\omega n; \quad \nabla N \approx \nabla n$$

(where n is small deviation of electron concentration from its equilibrium value), these equations can be written in the form

$$\mathbf{P} = (\varkappa Te/m\omega^2)\nabla n \tag{39}$$

$$-(e^2N/m\omega^2)\mathbf{E}; \; en + \operatorname{div}\mathbf{P} = 0.$$

Moreover, in writing these formulas, the well known definition of the polarization vector was used, $\mathbf{P} = eN\mathbf{r}$. In addition, taking into account (36c), let us write the relationship between P and E in the following coordinate form

$$P_x = -(e^2N/m\omega^2)E_x; \tag{40}$$

$$P_y = \frac{\varkappa T}{m\omega^2 4\pi}\left(\frac{\partial^2 E_y}{\partial y^2} + \frac{\partial^2 E_z}{\partial y\,\partial z}\right) - \frac{e^2N}{m\omega^2}E_y;$$

$$P_z = \frac{\varkappa T}{m\omega^2 4\pi}\left(\frac{\partial^2 E_y}{\partial y\partial z} + \frac{\partial^2 E_z}{\partial z^2}\right) - \frac{e^2N}{m\omega^2}E_z.$$

It is not difficult to see that the system (36), in combination with (40) resolves itself into two independent equation systems. It appears then that the wave with components E_x, H_y, H_z satisfies the same equations as in the problem considered in Sec. 1. The calculation of electron pressure results in changes of the second and third equations (36b) only:

$$\frac{\partial H_x}{\partial z} = ik_0\left[\varepsilon E_y + \frac{\beta^2}{k_0^2}\left(\frac{\partial^2 E_y}{\partial y^2} + \frac{\partial^2 E_z}{\partial y\,\partial z}\right)\right]; \tag{41}$$

$$-\frac{\partial H_x}{\partial y} = ik_0\left[\varepsilon E_z + \frac{\beta^2}{k_0^2}\left(\frac{\partial^2 E_y}{\partial y\,\partial z} + \frac{\partial^2 E_z}{\partial z^2}\right)\right],$$

where $\epsilon = 1-4\pi e^2 N/m\omega^2$ is the usual dielectric constant of the plasma (the absorption is neglected) and $\beta^2 = k_0^2\varkappa T/m\omega^2$. The parameter

$$\beta = (1/c)\sqrt{\varkappa T/m} \sim v/c \tag{42}$$

has an order of magnitude equal to the ratio of the electron thermal velocity to the velocity of light and represents under ordinary ionospheric conditions a rather small quantity; for instance, at temperatures of the order of 500° K: $\beta = 3\times 10^{-4}$.

Let us look for the solution in the form

$$H_x = w(z)e^{ik_0qy}, \quad E_z = u(z)e^{ik_0qy},$$

$$E_y = v(z)e^{ik_0qy}.$$

Then the system (41) with the elimination of E_y is reduced to the following two interrelated differential equations of the second order:

$$\frac{d^2w}{dz^2} - \frac{d\varepsilon/dz}{\varepsilon - \beta^2q^2}\frac{dw}{dz} \tag{43}$$

$$+ k_0^2(\varepsilon - q^2)w = \beta^2 q\frac{d\varepsilon/dz}{\varepsilon - \beta^2q^2}\frac{du}{dz},$$

$$(\beta^2 d^2u/dz^2) + k_0^2(\varepsilon - \beta^2q^2)u = qk_0^2(\beta^2 - 1)w.$$

Thus, the calculation of electron thermal motions results in equations of higher order.[9] The solutions of system (43) describe normal waves of two types, which in a few special cases, and also in the regions of small interaction, enable us to consider the wave field as the superposition of electromagnetic and plasma waves. Thus, for $q = 0$ (normal incidence) system (43) is resolved into two independent equations. The first equation gives the same results as Eq. (1) and the solution describes electromagnetic waves; the second gives a wave equation for plasma waves from which it is easy to obtain the known formula of the index of refraction for these waves [$n = (\epsilon/\beta)^{1/2}$] (see, for example, Ref. 1, Sec. 63). Besides, as the investigation of system (43) shows, the subdivision of the field into plasma and electromagnetic waves is possible in regions relatively far removed from the interaction region (vicinity of the point where $\epsilon = 0$).

It is easy to show that the solutions of system (43) will be analytical functions and a singularity appears in them when the small parameter β^2 approaches zero, β^2 being the coefficient of the highest order derivative in the fourth order equation equivalent to system (43).

We note that for small β^2 the first equation of system (43) differs little from Eq. (1), obtained without taking into account the thermal motions of electrons. The second equation for $\beta^2 \to 0$ turns into an algebraic relation

$$u(z) = -qw(z) / \varepsilon(z).$$

It can be shown by solving equations (43) by the method of successive approximations that the value of component E_z at the points where $\epsilon(z)$ becomes zero will have an order of magnitude equal to

$$|E_z|_{z=0} \approx qw(q)(\beta/\rho)^{-2/3}. \qquad (44)$$

The appearance of the multiplier $(\beta/\rho)^{-2/3}$ in formula (44) can be explained as follows. Let us write the equation for electrons moving under the influence of the E_z component by taking into account collisions and the pressure gradient (for processes dependent on time according to the law $e^{i\omega t}$)

$$-\omega^2 Nmr + i\omega\nu_{\text{eff}}\, Nmr = eNE_z + \varkappa T \partial N / \partial z. \qquad (45)$$

If, in addition, one assumes the continuity equation (38) and sets $\partial N/\partial z \approx kn$; $\partial N/\partial t \approx i\omega n$; div $N\mathbf{r} \approx kNr$, where $1/k$ is some characteristic dimension of the wave field, then Eq. (45) can be written in the form

$$-\omega^2 Nmr + i\omega\nu_{\text{eff}}\, Nmr = eNE_z + \varkappa T k^2 Nr,$$

from which, by comparing the terms $\omega\nu_{\text{eff}} Nmr$ and $\varkappa Tk^2 Nr$, it is easy to see that the quantity $(\varkappa T/m\omega^2)k^2$ plays the role of ν_{eff}/ω (by assuming collisions), on the assumption of the analogous effect of the pressure gradient (of the plasma waves).

Consequently, in the formula

$$|E_z| \approx qw(q) / |\alpha z + i\nu_{\text{eff}}/\omega| \qquad (46)$$

in the last case $(\varkappa T/m\omega^2)k^2$ will replace ν_{eff}/ω. Assuming, furthermore, that the dimension of the field inhomogeneity (by reducing the effect of plasma waves to some effective collision number) has an order of magnitude of $1/k$, we obtain the relationship

$$k = (m\omega^2\alpha / \varkappa T)^{1/3},$$

and the value of the field E_z at the resonance point ($z = 0$) will have an order of magnitude given by

$$|E_z| \approx qw(q) / (\varkappa T k^2 / m\omega^2) = qw(q)(\beta/\rho)^{-2/3}.$$

Consequently, the effect of plasma waves in our problem can be compared to the analogous effect of absorption, and some effective number of collisions can be introduced:

$$\nu_{\text{pl}} / \omega = (\beta/\rho)^{2/3}. \qquad (47)$$

For the data of E-layer ($\beta = 3\times10^{-4}$, $\alpha = 10^{-6}$, $\lambda_0 = 100$ m, $\rho = 2\pi\times10^2$) $\nu_{\text{pl}} \sim 10^3$.

For the F-layer ($\beta = 3\times10^{-4}$, $\lambda_0 = 30$m) we obtain $\nu_{\text{pl}} = 3.7\times10^2$ ($\alpha = 10^{-7}$); $\nu_{\text{pl}} = 1.7\times10^3$ ($\alpha = 10^{-6}$).

From these numbers it is seen that the calculation of the plasma wave effect is just as effective in some cases ($\alpha \sim 10^{-6}$) as the calculation of collisions ($\nu_{\text{eff}} \leq 10^3$). However, in the lower ($\nu_{\text{eff}} \geq 10^4$) ionospheric layers the absorption effect, connected with collisions, is predominant (See Ref. 1).

Finally, let us note that the assumption of nonlinear effects results in negligible corrections. Thus, in formula (46) along with ν_{eff}/ω there appears the magnitude $\alpha \int_0^t v_z\, dt$ where v_z is the speed of electrons in the z directions. This addition has an order of magnitude $\alpha e E_z / m\omega^2 \sim 10^{-5} E_z$ (for $\alpha = 10^{-7}$, $\omega = 2\pi\times10^7$ and is comparable to ν_{eff}/ω in the F-layer only for a field intensity $E_z \sim 1$ cgs unit at the resonance point. If the amplification coefficient of the field is taken to be 10^2, then at the boundary of the inhomogeneous layer the wave must have an intensity $E_z \sim 10^{-2}$ cgs unit of the order of several v/cm. Consequently, for ionospheric field intensities the role of nonlinear effects is negligible.

In conclusion I express deep gratitude to V. L. Ginzburg for suggesting the problem and for his help in the investigation.

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Quantum Field Theory with Causal Operators*

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A mathematical formalism is set up for the space-time description of field theory, starting from an action principle. It is proved that such a formalism necessarily leads to a representation by means of external sources.

1. INTRODUCTION

THE unsatisfactory state of quantum field theory makes urgent the task of investigating and reformulating the mathematical basis of the theory. Recently a number of papers have appeared, dealing with the space-time description of field theory. A consistently 4-dimensional treatment uses operators of a different type from those of the ordinary 3-dimensional formalism. The former type of operators we shall call "causal". They were introduced simultaneously by several authors authors,[1-3] working from different points of view.

Coester[1] found it necessary to define operators having the properties of causal operators, in order to construct a quantization scheme for Feynman amplitudes. Gol' fand considered these operators under the name of "quasi-fields," using them as a convenient device for calculating expectation-values of the S-matrix. The present author[2] defined causal operators as operators which refer to a definite direction of time, and which satisfy requirements connected with the principle of causality.

It has been proved[1-3] that the use of causal instead of ordinary field-operators does not change the results of the usual scattering-matrix theory. Equations for the state-vector were also introduced[1] into the causal operator formalism. But the form of these equations was fixed by requiring that the results of the causal operator theory should coincide with the results of the usual theory, and this cannot be regarded as a satisfactory basis for a logically constructed formalism.

The present paper contains a systematic development of the theory of causal operators, starting from an action principle. In Sec. 2 the action principle is formulated, and the basic difference between the causal operator theory and the usual theory is explained. The difference arises from our considering the situation from a 4-dimensional point of view. One consequence of this is the nonexistence of equations of motion for causal operators.* In Sec. 3 the state-vector is studied in a representation in which a complete set of commuting observables can only be constructed by means of causal operators. In this case the action principle leads to equations for the state vector which are identical (after some changes in notation) with the equations expressing the Schwinger functional in terms of external sources. The appearance of external sources seems to be inevitable in a space-time description of field theory. This seems to be connected with the inapplicability of a principle of stationary action. In Sec 4 the state-vector is studied by expressing it as a generalized Fock functional. The action principle leads to the equations for the state-vector which were proposed by Coester.[1] In Sec. 4 equations are also derived for Feynman amplitudes. In Sec. 5 the energy-momentum vector of the causal operator theory is defined.

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* This feature was not made clear in the author's second paper (see Ref. 2), where it was suggested that equations of motion of causal operators should exist. Hence, Eqs. (3) and (5) of Ref. 2 refer to the solution of a boundary-value problem, not for the causal operators, but for particular matrix elements of these operators. Equation (5), obtained by combining (3a) with (3b), is incorrect, since the matrix elements (3a) and (3b) refer to different functionals. In any case, Eq. (5) was never used in the rest of the paper.

2. STARTING EQUATIONS. THE ACTION PRINCIPLE.

The nucleon field operators are denoted by $\chi(x)$ and $\overline{\chi}(y)$, the meson field operators by $\Phi(x)$. We assume as already known that the causal operators $\chi(x)$, $\overline{\chi}(y)$ and $\Phi(z)$ anticommute or commute for all positions of the points x, y, z.

$$\{\chi(x), \overline{\chi}(y)\} = 0; \; [\Phi(x), \Phi(y)] = 0; \qquad [\chi(x), \Phi(z)] = 0, \text{etc.} \tag{2.1}$$

It is assumed that the matrixes χ, $\overline{\chi}$ and Φ are not diagonal in the particle occupation-numbers. We consider what consequences follow from the assumption that the basic field operators are χ, $\overline{\chi}$ and Φ with the properties (2.1).

In consequence of Eq. (2.1), we can construct a complete set of commuting observables ξ by means of the operators χ, $\overline{\chi}$ and Φ at the various points of space-time. Therefore, in the causal operator formalism, a state labelled by the set of eigenvalues ξ' has a well-defined space-time behavior. This fact is basic to the development of the theory of the operators χ, $\overline{\chi}$, Φ.

We suppose a complete set of state-vectors $\Omega(\xi')$ to be labelled by the eigenvalues ξ' of the set of observables ξ which constitute a space-time description of the system. The most general problem of the theory is to describe the state Ω of the interacting fields over the whole of space-time. Once Ω is known, the probabilties of all scattering processes and the properties of all stationary states are determined. The state-vector

$$\Omega = \int C(\xi')\, d\xi' \Omega(\xi') \tag{2.2}$$

will be fixed if the coefficients $C(\xi') - [\Omega(\xi'), \Omega]$ can be calculated. The coefficients $C(\xi')$ cannot be assigned arbitrarily, and the equations restricting their values follow from an action principle.

Consider an infinitesimal variation $\delta C(\xi')$ produced by infinitesimal variations of ξ' throughout space-time. The action principle is expressed by the equation

$$\delta C(\xi') = i\,(\Omega(\xi'),\ \delta W \cdot \Omega), \tag{2.3a}$$

where W is the action operator. This formulation of the principle is equivalent to the action principle of Feynman.[4] Since $\delta C(\xi') = [\delta\Omega(\xi'), \Omega]$, we can define an infinitesimal transformation operator $G_{\xi'}$, by*

$$(\delta\Omega(\xi'), \Omega) = (\Omega(\xi'), G_{\xi'} \Omega),$$

and then the action principle (2.3a) becomes

$$\delta_C \Omega \equiv G_{\xi'} \Omega = i\delta W \cdot \Omega. \tag{2.3b}$$

The variation $\delta_C \Omega$ in Eq. (2.3b) is the variation produced by varying the coefficients $C(\xi')$.

We assume that the action has the same form as in the usual theory, but expressed in terms of the operators χ, $\overline{\chi}$ and Φ. It is convenient to write the expression for W in the form of a double integral, with $\delta^4(x-y) = \delta(x-y)\,\delta(x_0 - y_0)$,

$$W = \int L(x, y)\, d^4x\, d^4y;$$

$$\begin{aligned} L(x, y) = {} & \frac{1}{2} i\delta^4(x-y) \\ & \times \{\overline{\chi}(y) D(x) \chi(x) - \chi(y) D(-x) \overline{\chi}(x) \\ & - {}^1\!/_2\, g\Phi(x) \gamma_5(y) [\overline{\chi}(y) \chi(x) - \chi(y) \overline{\chi}(x)]\} \\ & - \delta^4(x-y) \left[\left(\frac{\partial \Phi(x)}{\partial x_\nu}\right)^2 + \mu^2 \Phi^2(x)\right]. \end{aligned} \tag{2.4}$$

Here we use the notation

$$D(x) = i\,(\gamma_\lambda(x)\, \partial / \partial x_\lambda + m),$$

and $\gamma_\lambda(x)$ means that the matrix γ_λ operates on the spinor referring to the point x; thus,

$$\gamma_\lambda(x)\ \overline{\chi}(y) \chi(x) = \overline{\chi}(y) \gamma_\lambda \chi(x), \quad \text{etc.}$$

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* We always denote by P_μ the energy-momentum vector. To avoid confusion, the infinitesimal transformation operator is denoted by $G_{\xi'}$ instead of by the customary $P_{\xi'}$.

The representation (2.4) of the action W avoids the ambiguities which arise in considering products of causal operators referring to equal times. The operator $L(x, y)$ in Eq. (2.4) is symmetric in the coordinates x, y, and this is necessary in order that $L(x, y)$ should be invariant under charge-conjugation. The interaction term in $L(x, x)$ contains a normal product of nucleon field operators, in this formalism just as in the usual theory.

It is clear from Eq. (2.3b) that different choices of $\Omega\,(\xi')$ will in general correspond to different representations of the equations of motion for Ω. These representations reduce to two main types: (a) when the observables ξ are constructed directly from the operators χ, $\overline{\chi}$, Φ and (b) when the construction of the ξ involves the separation of the field-operators χ, $\overline{\chi}$, Φ into emission and absorption parts. In the first case the state-vector Ω is represented by a Schwinger functional,^{5} and in the second case by a generalized Fock functional.^{6}

3. THE STATE-VECTOR Ω AS A SCHWINGER FUNCTIONAL

If the operators ξ are constructed from the χ, $\overline{\chi}$ and Φ, then the variations $\delta\,\xi$ may be expressed in terms of operators $\delta\chi$, $\delta\overline{\chi}$, $\delta\Phi$. Then $G_{\xi'}$ In Eq. (2.3b) contains only these variations of the causal operators. We assume, as in the 3-dimensional formalism^7, that $\delta\chi$, $\delta\overline{\chi}$ and $\delta\Phi$ satisfy the relations

$$\{\delta\chi(x), \overline{\chi}(y)\} = 0; \tag{3.1}$$

$$\{\delta\overline{\chi}(x), \chi(y)\} = 0; \quad [\delta\Phi(x), \Phi(y)] = 0, \text{ etc.}$$

Since W is constructed from the operators χ, $\overline{\chi}$, Φ, Eq. (2.3b) is equivalent to

$$(G_\chi + G_{\overline{\chi}} + G_\Phi)\,\Omega \tag{3.2}$$

$$= i\,(\delta_\chi W + \delta_{\overline{\chi}} W + \delta_\Phi W)\,\Omega,$$

where G_χ, $G_{\overline{\chi}}$ and G_Φ are operators with the properties

$$[G_\chi, \chi] = \delta\chi; \ [G_{\overline{\chi}}, \overline{\chi}] = \delta\overline{\chi}; \ [G_\Phi, \Phi] = \delta\Phi, \tag{3.3}$$

and

$$\delta_\chi W = [G_\chi, W]; \ \delta_{\overline{\chi}} W = [G_{\overline{\chi}}, W]; \ \delta_\Phi W = [G_\Phi, W]$$

are the variations of W produced by the variations of χ, $\overline{\chi}$ and Φ.

To construct $G_{\xi'}$ we introduce operators $\pi(x)$, $\overline{\pi}(x)$ and $\Pi(x)$ determined by the following commutation rules:

$$\{\pi(x), \chi(y)\} = (1/i)\,\delta^4(x-y); \tag{3.4}$$

$$\{\overline{\pi}(x), \overline{\chi}(y)\} = i\delta^4(x-y);$$

$$[\Pi(x), \Phi(y)] = (1/i)\,\delta^4(x-y).$$

The operators π, $\overline{\pi}$ and Π are the 4-dimensional analogs of the canonically conjugate momenta in the usual 3-dimensional treatment. The operators G_χ, $G_{\overline{\chi}}$ and G_Φ then have the form

$$G_\chi = i\int \delta\chi(x)\,\pi(x)\,d^4x;$$

$$G_{\overline{\chi}} = -\,i\int \delta\overline{\chi}(x)\,\overline{\pi}(x)\,d^4x;$$

$$G_\Phi = i\int \delta\Phi(x)\,\Pi(x)\,d^4x.$$

Substituting these expressions into Eq. (3.2), using Eqs. (2.4) and (3.1), and remembering that the variations $\delta\chi$, $\delta\overline{\chi}$ and $\delta\Phi$ are independent, we obtain the system of equations for Ω

$$D\,(x) - g\gamma_5\Phi(x)]\,\chi(x)\,\Omega = i\overline{\pi}(x)\,\Omega; \tag{3.5}$$

$$[D(-x) - g\gamma_5(x)\,\Phi(x)]\,\overline{\chi}(x)\,\Omega = i\pi(x)\,\Omega;$$

$$\{i\,(\Box - \mu^2)\,\Phi(x) + {}^1\!/_2\, g\,[\overline{\chi}(y)\,\gamma_5\chi(x)$$

$$- \gamma_5\chi(y)\,\chi(x)]_{x=y}\}\,\Omega = i\Pi(x)\,\Omega.$$

A formal solution of Eq. (3.5) is easy to find if we assume that Ω can be represented in the form

$$\Omega \equiv \Omega\,[\chi, \overline{\chi}, \Phi] \tag{3.6}$$

$$= \sum_{n,m,k} \int F(x\ldots|\,y\ldots|\,z\ldots)$$

$$\times\,\chi(x)\ldots\overline{\chi}(y)\ldots\Phi(z)\ldots d^4x\ldots d^4y\ldots d^4z\ldots\,\Omega^0,$$

where Ω^0 satisfies the conditions

$$\pi(x)\,\Omega^0 = 0; \ \overline{\pi}(x)\,\Omega^0 = 0; \ \Pi(x)\,\Omega^0 = 0, \tag{3.7}$$

and the function $F(x\ldots|y\ldots|z\ldots)$ is antisymmetric in the coordinates $x\ldots,y\ldots$ and symmetric in the coordinates $z\ldots$. Then in Eq. (3.5) $\Phi(x)$ is to be considered as the operation of multiplying by a certain function $\Phi'(x)$ while $\Pi(x)$ is the operation of functional differentiation

$$\Pi(x)\,\Omega = -i\,(\delta\Omega/\delta\Phi'(x)).$$

In the same way $\pi(x)$ and $\overline{\pi}(x)$ in Eq. (3.5) are to be considered as functional differential operators, and they may be formally replaced by $-i\delta/\delta\chi(x)$ and $i\delta/\delta\overline{\chi}(x)$. Under these conditions, the formal solution of Eq. (3.5) is

$$\Omega[\chi,\overline{\chi},\Phi] = e^{iW}\Omega^0. \tag{3.8}$$

From Eq. (3.8) we can also obtain formal solutions for other cases.

The meaning of the state-vector Ω is most easily explained by examining the connection between Ω and the Schwinger functional Z which is a functional of external sources, $\eta(x)$, $\overline{\eta}(x)$ associated with the nucleon field, and $J(x)$ with the meson field. The functional Z is determined by the equations[5,8]

$$\left\{D(x) - ig\gamma_5\frac{\delta}{\delta J(x)}\right\}\frac{\delta Z}{\delta\overline{\eta}(x)} = -\eta(x)Z; \tag{3.9}$$

$$\left\{D(-x) - ig\gamma_5\frac{\delta}{\delta J(x)}\right\}\frac{\delta Z}{\delta\eta(x)} = \overline{\eta}(x)Z;$$

$$i(\Box - \mu^2)(\delta Z/\delta J(x)) = J(x)Z + gi\,\mathrm{Sp}\,\gamma_5(\delta^2 Z/\delta\eta(x)\,\delta\overline{\eta}(x)),$$

and the conditions

$$Z = 1,\ \delta Z/\delta\eta = \delta Z/\delta\overline{\eta} = \delta Z/\delta J = 0 \quad \text{for } J = \eta = \overline{\eta} = 0. \tag{3.10}$$

Comparing Eq. (3.9) with (3.5), we see that Eq. (3.9) is nothing else than the equation (3.5) for the state-vector $\Omega(\pi,\overline{\pi},\Pi)$, in the representation in which $\Pi(x)$ is the operation of multiplication by $\Pi'(x)$ and $\Phi(x)$ is the operation of functional differentiation

$$\Phi(x)\,\Omega[\pi,\overline{\pi},\Pi] = i\frac{\delta}{\delta\Pi'(x)}\Omega[\pi,\overline{\pi},\Pi].$$

The symbolic representation of $\chi(x)$ and $\overline{\chi}(x)$ is

$$\chi(x) = -i\delta/\delta\pi(x);\ \overline{\chi}(x) = i\delta/\delta\overline{\pi}(x).$$

The rôle of the external sources is thus played by the "conjugate momenta": $\pi(x)\sim\eta(x)$, $\overline{\pi}(x)\sim\overline{\eta}(x)$, $\Pi(x)\sim J(x)$. The initial conditions (3.7) for the functional $\Omega(\chi,\overline{\chi},\Phi)$ in the $\chi,\overline{\chi},\Phi$ representation correspond to the conditions (3.10) for the functional $\Omega(\pi,\overline{\pi},\Pi) = Z(\eta,\overline{\eta},J)$ in the $\pi,\overline{\pi},\Pi$ representation. Thus $\Omega(\pi,\overline{\pi},\Pi)$ is identical with the generating functional of the Green's functions for the system of interacting fields.

The appearance of external sources in the present formalism is at first glance unexpected, since the original action operator (2.4) does not involve external sources. But in a consistent space-time treatment the appearance of external sources is inevitable, as the following more general argument shows.

In the usual (either classical or quantized) 3-dimensional treatment of field theory, there exists a principle of stationary action. This principle states that variations of the field-quantities (the external sources being held constant), within the 4-dimensional volume bounded by hyperplanes t = constant in the remote past and future, do not change the action and do not influence the development of the system in time. In our 4-dimensional treatment of field theory, the development of the system in time is not considered, since the time-development is already included in the basic assumption of the 4-dimensional character of the wave-functional $\Omega(\xi)$. Since we assume the possibility of a 4-dimensional specification of a state, we necessarily assume that variations of the field-quantities (for example the variations $\delta\chi$, $\delta\overline{\chi}$ and $\delta\Phi$) at any point of space-time will influence the state-vector Ω. In other words, we are here speaking of variations which are forbidden from the point of view of the principle of stationary action. Hence, in the basic statement of the problem of giving a space-time treatment of field-theory, the principle of stationary action is necessarily abandoned. A particular consequence of this is the nonexistence of equations of motion for the field operators in the 4-dimensional treatment. There are no equations for χ, $\overline{\chi}$, Φ, but only equations such as Eq. (3.9) and (4.9) which restrict the possible choice of state-functionals.

Our mathematical formalism can then be equivalent to an ordinary 3-dimensional treatment, only if the 3-dimensional formalism allows variations which violate the principle of stationary action. Such variations are variations of external sources or of external parameters, and the space-time treatment is therefore necessarily and closely linked with the consideration of external sources and external invariant parameters.

From Eq. (2.3) we can in the usual way deduce the equation describing the development of the system as a particular external parameter is varied. For example, let the coupling constant g of the field interaction vary. Then the equation for $\Omega(g)$, which replaces the equation for the state-vector in the usual interaction representation, becomes

$$i\partial\Omega(g)/\partial g = W^0_{12}\Omega(g), \tag{3.11}$$

with W^0_{12} independent of g. $W_{12} = gW^0_{12}$ is the interaction term in the operator (2.4). Equation (3.11) has the formal solution

$$\Omega(g) \equiv S\Omega(0) = e^{iW_{12}}\Omega(0), \tag{3.12}$$

where $\Omega(0)$ is the functional describing noninteracting fields. Expressions of the form of Eq. (3.12) were obtained in several earlier investigations[1,2,3,8].

4. THE STATE-VECTOR AS A GENERALIZED FOCK FUNCTIONAL

We consider Eq. (2.3) in the representation in which the state-vector Ω is a generalized Fock functional.[6] We denote by $a^+_\rho(x)$, $b^+_\rho(x)$ the 4-dimensional creation operators of the nucleon field, and by $c^+(x)$ those of the meson fields, ρ being a spinor index. The commutation relations between these creation operators and the absorption operators $a_\rho(x)$, $b_\rho(x)$, $c(x)$, unlike the usual commutation relations, have on the right-hand side a 4-dimensional δ-function:

$$\{a_\rho(x), a^+_\sigma(y)\} = \delta_{\rho\sigma}\delta^4(x-y); \tag{4.1}$$

$$\{b_\rho(x), b^+_\sigma(y)\} = \delta_{\rho\sigma}\delta^4(x-y);$$

$$[c(x), c^+(y)] = \delta^4(x-y).$$

All other commutators vanish. The transformation properties of the operators $a(x)$, $b(x)$ and $c(x)$ are fixed by requiring that the quantities $[\Omega', a(x)\Omega]$, $[\Omega', b(x)\Omega]$ transform like a bispinor and a conjugate bispinor, while $[\Omega', c(x)\Omega]$ transforms like a pseudoscalar.*

The operators a, b, c and $a^+ b^+$, c^+ act on the functional Ω, which may be considered as a superposition of eigen-functionals of the number-operator of nucleons $a^+(x)\,a(x)d^4x$, the number-operator of antinucleons $\int b^+(x)\,b(x)\,d^4x$, and that of mesons $\int c^+(x)\,c(x)\,d^4x$:

$$\Omega = \sum \Omega_{nmk} = \sum_{nmk} (n!\,m!\,k!)^{-1/2} \tag{4.2}$$

$$\times \int d^4x_1 \ldots d^4x_n d^4y_1 \ldots d^4y_m d^4z_1 \ldots d^4z_k$$

$$\times f(x_1 \ldots x_n \,|\, y_1 \ldots y_m \,|\, z_1 \ldots z_k)$$

$$\times a^+(x_1) \ldots a^+(x_n)\, b^+(y_1) \ldots b^+(y_m)$$

$$\times c^+(z_1) \ldots c^+(z_k)\,\Omega_0.$$

Here Ω_0 is the normalized functional of the vacuum state, determined by the conditions $a(x)\,\Omega_0 = 0$, $b(x)\,\Omega_0 = 0$, $c(x)\,\Omega_0 = 0$, $(\Omega_0, \Omega_0) = 1$. The function $f(x_1 \ldots x_n \,|\, y_1 \ldots y_m \,|\, z_1 \ldots z_k)$, which plays the role of the coefficients $C(\xi')$ in Eq. (2.2), is the Feynman amplitude for n nucleons with coordinates $x_1, \ldots, x_n$, m antinucleons with coordinates $y_1, \ldots, y_m$, and k-mesons with coordinates $z_1, \ldots, z_k$. The function $f(x \ldots \,|\, y \ldots \,|\, z \ldots)$ is antisymmetric in the x and y coordinates, and symmetric in the z coordinates. Coester[1] introduced a functional Ω of the form of Eq. (4.2).

The scalar product (Ω, Ω') of the generalized functionals Ω and Ω' is defined in analogy to the three-dimensional case[6]. If $f(x \ldots \,|\, y \ldots \,|\, z \ldots)$ are the functions in the expansion (4.2) of Ω, and if

$$\Omega'^{+} = \sum_{nmk} \int \Omega_0^+\, c(z_k) \ldots b(y) \ldots \tag{4.3}$$

$$a(x)\, f'^{*}(x \ldots \,|\, y \ldots \,|\, z \ldots)$$

is the functional conjugate to Ω', then

$$(\Omega', \Omega) \tag{4.4}$$

$$= \sum_{nmk} \int f'^{*}(x_1 \ldots x_n \,|\, y_1 \ldots y_m \,|\, z_1 \ldots z_k)$$

$$\times f(x_1 \ldots x_n \,|\, y_1 \ldots y_m \,|\, z_1 \ldots z_k)$$

$$\times dx_1 \ldots dx_n dy_1 \ldots dy_m dz_1 \ldots dz_k.$$

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* One may call the operator $a(x)$ a bispinor, in the same sense in which the Dirac matrices γ_ν are sometimes said to constitute a vector.

To obtain the explicit form of Eq. (2.3) for a generalized Fock functional Ω, one must establish the correspondence between the creation and absorption operators and the causal operators χ, $\overline{\chi}$ and Φ. The correspondence is based on the supposition that the operators χ, $\overline{\chi}$ and Φ can be separated into creation parts χ^c, $\overline{\chi}^c$ and Φ^c and absorption parts χ^a, $\overline{\chi}^a$ and Φ^a, all the absorption parts commuting or anticommuting with each other, and all the creation parts likewise. This supposition is essential to the argument. It limits the possible form of the operators χ, $\overline{\chi}$ and Φ, and restricts the form of the commutation relations between the creation and absorption parts of these operators. From the conditions (2.1) and (4.1) we deduce that these commutation relations must be

$$\{\chi^a(x), \overline{\chi}^c(y)\} = \sigma_F(x, y); \tag{4.5}$$

$$[\Phi^a(x), \Phi^c(y)] = d_F(x, y);$$

$$\{\overline{\chi}^a(x), \chi^c(y)\} = -\sigma_F(y, x),$$

where the quantities σ_F and d_F are functions and not field-operators. The functions $\sigma_F(x, y)$ and $d_F(x, y)$ are given by

$$\sigma_F(x, y) = (\Omega_0, \chi(x)\overline{\chi}(y)\Omega_0)$$

$$= -(\Omega_0, \overline{\chi}(y)\chi(x)\Omega_0);$$

$$d_F(x, y) = (\Omega_0, \Phi(x)\Phi(y)\Omega_0)$$

$$= (\Omega_0, \Phi(y)\Phi(x)\Omega_0).$$

These equations show that σ_F and d_F are Green's functions of causal, or Feynman, type.

The commutation relations (4.3) will be obeyed if we assume $\chi^a(x) \equiv a(x)$, $\overline{\chi}^a(x) \equiv b(x)$, $\Phi^a(x) \equiv c(x)$, and represent the operators χ, $\overline{\chi}$ and Φ in the form

$$\chi(x) = a(x) - \int \sigma_F(x, y)\, b^+(y)\, d^4y; \tag{4.6}$$

$$\overline{\chi}(x) = b(x) + \int a^+(y)\, \sigma_F(y, x)\, d^4y;$$

$$\Phi(x) = c(x) + \int d_F(x, y)\, c^+(y)\, d^4y.$$

This representation of the causal operators was already given by Coester[1]. The choice of the functions σ_F and d_F, with the operators χ, $\overline{\chi}$ and Φ given by Eq. (4.6), fixes the meaning of the basic states to which the functionals Ω_{nmk} are referred. Usually d_F and σ_F are referred to the noninteracting fields, in which case $d_F = \frac{1}{2}\Delta_F$ and $\sigma_F = -\frac{1}{2}S_F$; however, renormalization is then impossible. To obtain a renormalizable theory one must choose[9] $\sigma_F = -\frac{1}{2}S'_F$ and $d_F = \frac{1}{2}\Delta'_F$.

Remembering that

$$f(x_1 \ldots x_n \mid y_1 \ldots y_m \mid z_1 \ldots z_k) \tag{4.7}$$

$$= (n!\; m!\; k!)^{-1/2} (\Omega_0, c(z_k) \ldots$$

$$c(z_1)\, b(y_m) \ldots b(y_1)\, a(x_n) \ldots a(x_1)\, \Omega),$$

we find the expression for an infinitesimal transformation operator

$$G = -\int [\delta a(x)\, a^+(x) \tag{4.8}$$

$$+ \delta b(x)\, b^+(x) + \delta c(x)\, c^+(x)]\, d^4x.$$

Since the variations $\delta a(x)$, $\delta b(x)$ and $\delta c(x)$ are independent, the action principle (2.3) is equivalent to the following system of equations for Ω:

$$\{D(x)\chi(x) - g\gamma_5\Phi(x)\chi(x)\}\Omega = -b^+(x)\Omega; \tag{4.9}$$

$$\{D(-x)\overline{\chi}(x) - g\Phi(x)\overline{\chi}(x)\gamma_5\}\Omega = a^+(x)\Omega;$$

$$\{i(\Box - \mu^2)\Phi(x) + gN[\overline{\chi}(x)\gamma_5\chi(x)]\}\Omega = -c^+(x)\Omega.$$

These equations were originally proposed by Coester[1].

Equations (4.9) imply an infinite system of coupled equations for the functions $f(x\ldots|y\ldots|z\ldots)$. These coupled equations are a generalization of the equations obtained in Fock's method of functionals[6]. They differ from Fock's equations in two ways: first, not only the meson field but also the nucleon field is treated by means of second quantization; second, the amplitudes are Feynman amplitudes of four-dimensional type*. Inserting the expansion (4.2) of the functional Ω into the first equation (4.9), we obtain the equations for the coefficients $f(x_1\ldots x_n|y_1\ldots y_m|z_1\ldots z_k)$ with $(k, n, m = 0, 1, 2, \ldots\infty)$,

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* The equations for the lowest Feynman amplitudes were obtained by Zimmerman (see Ref. 9) without using a four-dimensional method of quantization.

$$\begin{aligned}
&\sqrt{n+1}\, D(x)\, f(xx_1 \ldots x_n \mid y_1 \ldots y_m \mid z_1 \ldots z_k) \\
&\quad - \frac{(-1)^n}{\sqrt{m}} [\{D(x)\, \sigma_F(x, y_1) \\
&- \delta^4(x-(y_1)\} f(x_1 \ldots x_n \mid y_2 \ldots y_m \mid z_1 \ldots z_k)]_y \\
&\quad = g\gamma_5(x) \{\sqrt{n+1}\sqrt{k+1} \\
&\times f(xx_1 \ldots x_n \mid y_1 \ldots y_m \mid xz_1 \ldots z_k) \\
&\quad -(-1)\, \frac{\sqrt{k+1}}{\sqrt{m}} \\
&\times [\sigma_F(x, y_1)\, f(x_1 \ldots x_n \mid y_2 \ldots y_m \mid xz_1 \ldots z_k)]_y \\
&+ [f(xx_1 \ldots x_n \mid y_1 \ldots y_m \mid z_2 \ldots z_k)\, d_F(x, z_1)]_z \\
&\quad - \frac{(-1)^n}{\sqrt{mk}} [f(x_1 \ldots x_n \mid y_2 \ldots y_m \mid z_2 \ldots z_k) \\
&\qquad \times \sigma_F(x, y_1)\, d_F(x, z_1)]_{y,z}\}.
\end{aligned} \tag{4.10}$$

The factor $(-1)^n$ arises from the fact that the operator $b^+(y)$ must be computed through n operators a in order to arrive at the beginning of the b^+ operators. The notation $[\ \ ]_{y,z}$ means that the expression inside the brackets must be symmetrized in the meson coordinates (z) and antisymmetrized in the antinucleon coordinates (y); for example,

$$\begin{aligned}
&[\sigma_F(x, y_1)\, f(\ldots \mid y_2 \ldots y_m \mid \ldots)]_y \\
&\quad = \sigma_F(x_1 y_1)\, f(\ldots \mid y_2 \ldots y_m \mid \ldots) \\
&- \sigma_F(x, y_2)\, f(\ldots \mid y_1 y_3 \ldots y_m \mid \ldots) \\
&\quad + \sigma_F(x, y_3)\, f(\ldots \mid y_1 y_2 y_4 \ldots y_m \mid \ldots) - \ldots].
\end{aligned}$$

The system of equations for $f(x \ldots \mid y \ldots \mid z \ldots)$ arising from the second equation (4.9) is

$$\begin{aligned}
&(-1)^n \sqrt{m+1} \\
&\quad \times D(-y)\, f(x_1 \ldots x_n \mid yy_1 \ldots y_m \mid z_1 \ldots z_k) \\
&\quad + \frac{1}{\sqrt{n}} [\{D(-y)\, \sigma_F(x_1, y) \\
&- \delta^4(x_1 - y)\} f(x_2 \ldots x_n \mid y_1 \ldots y_m \mid z_1 \ldots z_k)]_x \\
&\quad = g\gamma_5(y) \{\sqrt{m+1}\sqrt{k+1}\, (-1)^n \\
&\times f(x_1 \ldots, x_n \mid yy_1 \ldots y_m \mid yz_1 \ldots z_k) \\
&\quad + (-1)^n [d_F(y, z_1) \\
&\quad f(x_1 \ldots x_n \mid yy_1 \ldots y_m \mid z_2 \ldots z_k)]_z \\
&\quad + n^{-1/2} [\sigma_F(x_1, y) \\
&\times f(x_2 \ldots x_n \mid y_1 \ldots y_m \mid yz_1 \ldots z_k)]_x \\
&\quad + (nk)^{-1/2} [d_F(y, z_1) \\
&\times f(x_2 \ldots x_n \mid y_1 \ldots y_m \mid z_2 \ldots z_k)\, \sigma_F(x_1 - y)]_{x,z}\}.
\end{aligned} \tag{4.11}$$

The system of equations for $f(x \ldots \mid y \ldots \mid z \ldots)$ equivalent to the third equation (4.9) is

$$\begin{aligned}
&i(\Box_z - \mu^2) \\
&\times f(x_1 \ldots x_n \mid y_1 \ldots y_m \mid zz_1 \ldots z_k) \\
&\times \sqrt{k+1} + \frac{1}{\sqrt{k}} [\{i(\Box_z - \mu^2)\, d_F(z, z_1) \\
&- \delta^4(z - z_1)\} f(x_1 \ldots x_n \mid y_1 \ldots y_m \mid z_2 \ldots z_k)]_z \\
&\quad = g\{\sqrt{m+1}\sqrt{n+1}\, \gamma_5(z) \\
&\times f(zx_1 \ldots x_n \mid zy_1 \ldots y_m \mid z_1 \ldots z_k)(-1)^n \\
&\quad - (-1)^n (nm)^{-1/2} [\sigma_F(x_1, z)\gamma_5(z)\, \sigma_F(z, y_1) \\
&\quad \times f(x_2 \ldots x_n \mid y_2 \ldots y_m \mid z_1 \ldots z_k)]_{x,y} \\
&\quad + [\sigma_F(x_1, z)\, \gamma_5(z) \\
&\times f(z_2 x_2 \ldots x_n \mid y_1 \ldots y_m \mid z_1 \ldots z_k)]^*_x \\
&\quad + [\gamma_5(z)\, \sigma_F(z, y_1) \\
&\times f(x_1 \ldots x_n \mid zy_2 \ldots y_m \mid z_1 \ldots z_k)]^*_y\},
\end{aligned} \tag{4.12}$$

where $[\ \ ]^*_x$ and $[\ \ ]^*_y$ mean that the expression in the first bracket must be antisymmetrized in the coordinates $(zx_1 \ldots x_n)$, and the expression in the second bracket must be antisymmetrized in $(zy_1 \ldots y_m)$.

Equations (4.10), (4.11) and (4.12) have a very clumsy appearance. They can be simplified a little by suitable choice of the functions $\sigma_F(x, y)$ and $d_F(x, y)$. Compared with the functions $T(x \ldots \mid y \ldots \mid z \ldots)$ for which equations were derived by Zimmerman[9] and others[10], the functions $f(x \ldots \mid y \ldots \mid z \ldots)$ have the advantage of referring to an orthogonal system of states. A detailed investigation of Eqs. (4.10)-(4.12) will be published separately.

5. THE ENERGY-MOMENTUM VECTOR

In the usual three-dimensional treatment of quantum field theory, the expression for the energy-momentum vector is derived from the Lagrangian, and the identification of the energy-momentum vector as a displacement operator is a consequence of the canonical commutation relations. In our four-dimensional treatment of quantum field theory there is no variation principle, no canonical commutation relations, and no equations of motion of the field-operators. We therefore define the energy-momentum vector P_μ ($P_4 = iP_0$) as a quantity possessing the properties of a displacement operator,

$$-i\,[P_\mu, a(x)] = \partial a(x)/\partial x_\mu; \tag{5.1}$$

$$-i\,[P_\mu, a^+(x)] = \partial a^+(x)/\partial x_\mu;$$

$$-i\,[P_\mu, b(x)] = \partial b(x)/\partial x_\mu;$$

$$-i\,[P_\mu, c^+(x)] = \partial c^+(x)/\partial x_\mu,$$

etc., and with components commuting with one another, $[P_\mu, P_\nu] = 0$. The vacuum state Ω_0 must satisfy the condition

$$P_\mu \Omega_0 = 0. \tag{5.2}$$

We determine the operator P_μ by considering the variation $\delta_{(\mu)}\Omega$ produced in any state-vector (4.2) by variations of the operators a^+, b^+, c^+ as a result of an infinitesimal displacement of coordinates $x_\mu \to x_\mu + \delta x_\mu^0$. The displacement δx_μ^0 is independent of the point x. The variation $\delta_{(\mu)}\Omega$ will be equal to

$$\delta_{(\mu)}\Omega = -iP_\mu \delta x_\mu^0\,\Omega.$$

The energy-momentum tensor which satisfies the conditions (5.1) and (5.2) is, by virtue of Eq. (4.2),

$$P_\mu = -i\int\Big\{a^+(x)\,\frac{\partial a(x)}{\partial x_\mu} + b^+(x)\,\frac{\partial b(x)}{\partial x_\mu} + c^+(x)\,\frac{\partial c(x)}{\partial x_\mu}\Big\}\,d^4x. \tag{5.3}$$

In momentum-space this becomes

$$P_\mu = \int\{a^+(q)\,a(q) + b^+(q)\,b(q) + c^+(q)\,c(q)\}\,q_\mu d^4q.$$

Here q_0 is an independent variable.

It is important to distinguish the variation $\delta_{(\mu)}\Omega$ from the total variation $\delta_\mu\Omega$ connected with the displacement $x_\mu \to x_\mu + \delta x_\mu^0$. The total variation $\delta_\mu\Omega$ is zero, because the variation $\delta_\mu W$ of the action is zero then the coordinates are displaced, by virtue of the conservation law for energy and momentum [see Eq. (2.3)]. To obtain $\delta_\mu\Omega$ explicitly, one must first consider the state-vector (4.2) and the action (2.4), with each four-dimensional integration extending not over the whole space-time but over a volume bounded by two hypersurfaces σ_1 and σ_2. The limiting process which extends the integration to the whole space-time is only to be performed afterwards. Then $\delta_\mu\Omega$ and $\delta_\mu W$ refer to a rigid displacement of σ_1 and σ_2 through the vector δx_μ. The equation $\delta_\mu\Omega = 0$ means that the derivatives of the integrand of Eq. (3.2) with respect to every coordinate x_μ, y_μ, z_μ are zero. Therefore, the effect of the operator P_μ on the state-vector Ω is to differentiate the amplitudes $f(x\ldots|y\ldots|z\ldots)$. If $P_\mu\Omega = \Omega'$, and if $f'(x\ldots|y\ldots|z\ldots)$ are the amplitudes of the functional Ω', then

$$f'(x_1\ldots x_n \mid y_1\ldots y_m \mid z_1\ldots z_k) = -i\Big[\sum_n \frac{\partial}{\partial x_\mu} + \sum_m \frac{\partial}{\partial y} + \sum_k \frac{\partial}{\partial z_\mu}\Big] \times f(x_1\ldots x_n \mid y_1\ldots y_m \mid z_1\ldots z_k). \tag{5.4}$$

The meaning of P_μ as an energy-momentum vector can easily be verified by considering the example of noninteracting fields. In the special case of a free nucleon field, for example, we find by using the equation of motion (4.7)

$$i\int a^+(x)\,\frac{\partial a(x)}{\partial x_0}\,d^4x\Omega = \int a^+(x)\,[\alpha\mathbf{p} + \gamma_4 m]\,a(x)\,\Omega.$$

The operator Q, representing the total charge of the nucleon field, is obtained, just as in the usual three-dimensional treatment, by considering an infinitesimal phase transformation of the causal operators. The result is

$$Q = \int[a^+(x)\,a(x) - b^+(x)\,b(x)]\,d^4x. \tag{5.5}$$

CONCLUSION

The theory of the operators χ, χ and Φ is a consistent space-time formulation of the quantum theory of fields, closely connected with the usual formulation of field theory with external sources. But there is an important difference between the roles played by the external sources in the two theories. In the usual three-dimensional theory, the external sources $\eta(x)$, $\bar{\eta}(x)$, $J(x)$ are auxiliary quantities, external to the theory itself. In the four-dimensional treatment, these quantities are derived from the fundamental operators $\chi(x)$, $\bar{\chi}(x)$, $\Phi(x)$ and from the creation and absorption operators a, b, c, a^+, b^+, c^+. This may be considered an advantage of the four-dimensional theory.

One may expect that the mathematical formalism of the operators χ, $\bar{\chi}$ and Φ will be suitable for constructing field theories without using representations which refer to "bare" particles. Lehmann, Zimmermann and Symanzik[11] attempted to construct a theory of this kind. Starting from general conditions of invariance, causality and asymptotic behavior of field operators at $t = \pm\infty$, they obtained a variety of equations connecting matrix elements of T-products. In the space-time treatment, these equations can be derived very easily by using the representation in which the state-vector is a Fock functional. As an example we give the derivation of the reduction formula which occupies a central position in the work of Lehmann *et al.*[11]. By definition,

$$T(x \ldots | y \ldots | z \ldots) \tag{6.1}$$

$$= (\Omega_0, \chi(x) \ldots \bar{\chi}(y) \ldots \Phi(z) \ldots \Omega(g))$$

$$= (\Omega_0, \chi(x) \ldots \bar{\chi}(y) \ldots \Phi(z) \ldots . S\Omega(0)),$$

where $\Omega(0)$ is a functional of the noninteracting fields, and S is the scattering matrix [see Eq. (3.12)], the detailed form of which is irrelevant. The important point is that S commutes with χ, $\bar{\chi}$ and Φ. We assume for simplicity that $\Omega(0)$ refers to a state of one nucleon with wave-function $f_1^0(x')$ and one meson with wave-function $f_3^0(z')$. Then Eqs. (4.2) and (4.9) give

$$\Omega(0) = \int f_1^0(x') f_3^0(z') a^+(x') c^+(z') d^4x' d^4z' \, \Omega_0 \tag{6.2}$$

$$= \int D(x') \chi(x')$$

$$\times i(\Box_{z'} - \mu^2) \Phi(z') \Omega_0 f_1^0(x') f_3^0(z') d^4x' d^4z'$$

and

$$T(x \ldots | y \ldots | z \ldots) \tag{6.3}$$

$$= \int D(x') i(\Box_{z'} - \mu^2)(\Omega_0, \chi(x)$$

$$\ldots \bar{\chi}(y) \ldots \Phi(z) \ldots \chi(x')$$

$$\times \Phi(z') S\Omega_0) f_1^0(x') f_3^0(z') d^4x' d^4z'.$$

This is the required reduction formula[11]. The asymptotic condition is here equivalent to the requirement that the wave-functions $f_1^0(x')$ and $f_3^0(z')$ are composed of positive-energy components only.

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Anomalous Magnetic Moments of Nucleons

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Calculations are presented for the anomalous magnetic moment of nucleons in which the excited nucleon states with spin 3/2 and isotopic spin 3/2 are included. Divergent expressions were obtained which were regularized with the aid of Feynman factors. The cut-off parameter can be so selected that agreement between theory and experiment is obtained.

1. INTRODUCTION

THE usual theory for the interaction of π-mesons with nucleons explains qualitatively the anomalous magnetic moment of nucleons. Pseudoscalar meson theory gives the correct sign for the magnetic moment of the proton and neutron but does not give quantitative agreement with experimental values. For the case of a mixed pseudoscalar and pseudovector coupling of the meson and nucleon fields in a symmetric meson theory one obtains the following expressions for the anomalous moment[1,2]:

$$\delta\mu_p = \frac{e}{2m}\frac{g^2(4/\Lambda + s^2)}{2\pi^2} \times \left[\frac{1}{4} - \frac{3\Lambda}{2} + \frac{5\Lambda - 3\Lambda^2}{4}\ln\frac{1}{\Lambda} - \frac{\Lambda^{1/2}(4 - 11\Lambda + 3\Lambda^2)}{2(4-\Lambda)^{1/2}}\cos^{-1}\frac{\Lambda^{1/2}}{2}\right] \tag{1}$$

$$\delta\mu_N = -\frac{e}{2m}\frac{g^2(4/\Lambda + s^2)}{2\pi^2} \times \left[1 + \frac{\Lambda}{2}\ln\frac{1}{\Lambda}\frac{\Lambda^{1/2}(2-\Lambda)}{(4-\Lambda)^{1/2}}\cos^{-1}\frac{\Lambda^{1/2}}{2}\right]. \tag{2}$$

Here gs/μ is the pseudovector and g is the pseudoscalar coupling constant, μ is the meson mass. It follows immediately that

$$|\delta\mu_N / \delta\mu_p| \approx 7. \tag{3}$$

Experimental measurements of the value of the anomalous magnetic proton and neutron moment yield

$$\delta\mu_p^{\text{exp.}} = 1.79\ \mu_0; \quad \delta\mu_N^{\text{exp}} = -1.91\ \mu_0, \tag{4}$$

where μ_0 is the nuclear magneton (in units of $c = 1$, $\hbar = 1$). Consequently,

$$|\delta\mu_N^{\text{exp}} / \delta\mu_p^{\text{exp}}| \approx 1. \tag{5}$$

Thus, a quantitative agreement between theory and experiment is absent.

Progress in the use of isobaric theory of nucleons in problems of pion-nucleon scattering[3] and gamma-pion production[4] raises the need for the consideration of the effects of excited nucleon states with spin 3/2 and isotopic spin 3/2 on the magnetic moments of nucleons. The object of the present paper is to calculate the anomalous magnetic moment of nucleons with the inclusion of these excited states. The calculation is based on a semi-phenomenological theory of π-meson and nucleon interactions, as developed by Tamm, Gol'fand and Fainberg[3]. All quantities are written in Feynman's notation[5]. For clarity, expressions of the form $a_\mu\gamma_\mu$ are indicated by the sign $\hat{}$ i.e., $\hat{a} \equiv a_\mu\gamma_\mu$.

2. LAGRANGIAN SYSTEM. EQUATIONS OF MOTION

The Lagrangian system for nucleons and mesons in an electromagnetic field in symmetric pseudoscalar meson theory with mixed pseudoscalar and pseudovector coupling between the mesonic and nucleonic fields has the form

$$L = L_0 + L' \tag{6}$$

$$L_0 = \bar{\psi}(i\hat{\nabla} - m)\psi - \bar{B}_\mu(i\hat{\nabla} - M_1)B_\mu \left(\bar{B}_\mu\frac{\partial D}{\partial x_\mu} - \frac{\partial \bar{D}}{\partial x_\mu}B_\mu\right) - \frac{3}{2}i\bar{D}\hat{\nabla}D - 3M_1\bar{D}D + \frac{1}{2}\left[\left(\frac{\partial\varphi_i}{\partial x_\nu}\right)^2 - \mu^2\varphi_i^2\right];$$

$$L' = -e\bar{\psi}\frac{1+\tau_3}{2}\hat{A}\psi + e\bar{B}_\mu Q\hat{A}B_\mu + \frac{g\sqrt{4\pi}}{\mu}\bar{\psi}\left(\gamma_\nu\frac{\partial\varphi_i}{\partial x_\nu} + i\mu s\varphi_i\right)\tau_i\gamma_5\psi + \frac{g_1\sqrt{4\pi}}{\mu}(\bar{\psi}S_iB_\nu + \bar{B}_\nu S_i^+\psi)\frac{\partial\varphi_i}{\partial x_\nu} + \frac{ie\varepsilon}{m}(\bar{\psi}\gamma_\nu\gamma_5 f_{\nu\mu}NB_\mu - \bar{B}_\mu\gamma_\nu\gamma_5 f_{\nu\mu}N^+\psi) + eA_\nu\left(\varphi_1\frac{\partial\varphi_2}{\partial x_\nu} - \varphi_2\frac{\partial\varphi_1}{\partial x_\nu}\right)$$

(repeated Greek indices imply a summation from 1 to 4 and for Latin indices, 1 to 3) where τ_i is the isotopic spin matrix of the nucleon, Q is the charge operator of the isobar (isotopic spin operator) $(1+\tau_3)/2$ is the charge operator of the nucleon, viz.,

$$Q = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \qquad \frac{1+\tau_3}{2} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \tag{7}$$

S_i are the matrices introduced in Refs. 3 and 6; they change the charge state of a nucleon from the ordinary to the isobar and back; ϵ is the constant for the additional interaction of the nucleon with the electromagnetic field which is responsible for transitions from the ordinary to the isobaric state and back; N is an operator which accomplishes these transitions in isotopic space. Its form and the form of ϵ is given by Ritus[4].

Applying the variational principle with the auxiliary condition that $\bar{B}_\mu \gamma_\mu = 0$, we obtain the following equations for the wave functions ψ and B_μ:

$$\begin{aligned} (i\hat{\nabla} - m)\psi = \Big[& e\frac{1+\tau_3}{2}\hat{A}_l \\ & - \frac{g\sqrt{4\pi}}{\mu}\Big(\gamma_\nu \frac{\partial \varphi_i}{\partial x_\nu} + i\mu s \varphi_i\Big)\tau_i\gamma_5\Big]\psi \\ & - \Big[\frac{g_1\sqrt{4\pi}}{\mu} S_i \frac{\partial \varphi_i}{\partial x_\nu} + i\frac{e\varepsilon}{m}\gamma_\mu\gamma_5 f_{\mu\nu} N\Big] B_\nu; \\ (i\hat{\nabla} - M_1) B_\mu - \frac{i}{2}\gamma_\mu \frac{\partial B_\nu}{\partial x_\nu} & = eQ\Big[\hat{A}\delta_{\mu\nu} - \frac{1}{2}\gamma_\mu A_\nu\Big] B_\nu \\ & + \Big[\frac{g_1\sqrt{4\pi}}{\mu}\Big(\frac{\partial \varphi_i}{\partial x_\mu} - \frac{1}{4}\gamma_\mu\gamma_\nu \frac{\partial \varphi_i}{\partial x_\nu}\Big) S_i^+ \\ & - \frac{ie\varepsilon}{m}\Big(\gamma_\nu\gamma_5 f_{\nu\mu} - \frac{1}{4}\gamma_\mu\gamma_\sigma\gamma_\tau\gamma_5 f_{\tau\sigma}\Big) N^+\Big]\psi \\ - \Big(\frac{\partial^2}{\partial x_\mu^2} + \mu^2\Big)\varphi_i & = 2eA_\nu e_{iS}\frac{\partial \varphi_S}{\partial x_\nu}. \end{aligned} \tag{8}$$

The symbol e_{iS} is specified by the identities

$$-e_{12} = e_{21} = -1, \quad e_{11} = e_{22} = 0.$$

Eq. (8) can be written in the form

$$\sum_{\beta=0}^{4} L_{\alpha\beta}\Phi_\beta = \sum_{\beta=0}^{4} A_{\alpha\beta}\Phi_\beta; \quad \alpha = 0, 1, 2, 3, 4.$$

$$\Phi_0 = \Psi; \quad \Phi_\mu = B_\mu; \quad \mu = 1, 2, 3, 4;$$

$$\begin{aligned} A_{00} = {} & e\frac{1+\tau_3}{2}\hat{A} \\ & - \frac{g\sqrt{4\pi}}{\mu}\Big(\gamma_\nu \frac{\partial \varphi_i}{\partial x_\nu} + i\mu s\varphi_i\Big)\tau_i\gamma_5; \\ A_{0\mu} = {} & -\Big(\frac{g_1\sqrt{4\pi}}{\mu} S_i \frac{\partial \varphi_i}{\partial x_\mu} + i\frac{e\varepsilon}{m}\gamma_\nu\gamma_5 f_{\nu\mu} N\Big); \\ A_{\mu 0} = {} & \frac{g_1\sqrt{4\pi}}{\mu}\Big(\frac{\partial \varphi_i}{\partial x_\mu} - \frac{1}{4}\gamma_\mu\gamma_\nu \frac{\partial \varphi_i}{\partial x_\nu}\Big) S_i^+ \\ & - \frac{ie\varepsilon}{m}\Big(\gamma_\nu f_{\nu\mu} - \frac{1}{4}\gamma_\mu\gamma_\sigma\gamma_\tau f_{\tau\sigma}\Big)\gamma_5 N^+. \end{aligned} \tag{9}$$

The system (8) can be solved by the Feynman method with the help of the inverse operators $(L^{-1})_{\alpha\beta} = K_{\alpha\beta}$, where

$$\begin{aligned} K_{00} = {} & (\hat{p} - m)^{-1}; \\ K_{\mu\nu} = {} & (\hat{p} - M_1)^{-1}[\delta_{\mu\nu} \\ & - (1/_6 M_1^2)(2\gamma_\mu\hat{p} + \hat{p}\gamma_\mu + 3M_1\gamma_\mu)p_\nu]; \\ & K_{0\mu} = K_{\mu 0} = 0. \end{aligned} \tag{10}$$

3. CALCULATIONS OF MATRIX ELEMENTS

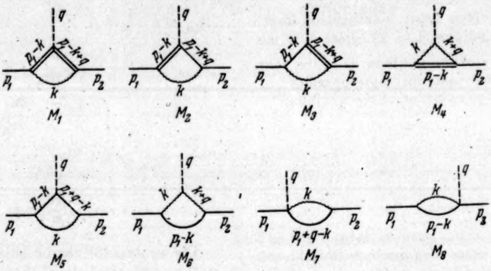
One must calculate the contribution to the anomalous magnetic moment of nucleons of processes whose corresponding diagrams are illustrated in the Figure.

In view of the strong singularity of the inverse operator $K_{\mu\nu}$, the matrix elements M diverge rapidly (4th order divergence). This divergence can be removed by the introduction of the Feynman cut-off factors.

As a simplification in the calculations, the cut-off is not applied to all matrix elements at once but to each component separately, i.e., for each divergence of degree n in the meson momentum, i.e., k^n, one needs a separate cut-off factor $[-\lambda^2/(k^2 - \mu^2 - \lambda^2)]^{N_n}$, where N_n is the smallest integer for which the regularization integral for the given degree k^n is finite. This procedure corresponds to the fact that the cut-off does not change the convergence in the absence of a cut-off value, i.e., the convergence for $\lambda = \infty$.

The very unwieldy expressions in the integrand for M are transformed in a direct fashion: the quantities $K_{\mu\nu}$, K_{00}, $S_{\nu\tau}$, Q_σ are intermultiplied and the results grouped according to powers of k; denominators of the form

$$(k^2 - 2kp_1 - \Delta_1)(k^2 - 2kp_2 - \Delta_2)^{-1}$$



Diagrams of third-order contributions to the anomalous magnetic moment of nucleons. In all diagrams, $p_2 = p_1 + q$; – – – = external electromagnetic field; ——— = nucleon in the ordinary state; ═══ = nucleon in isobaric state; ∿∿∿ = meson.

are transformed by the Feynman formula

$$\frac{1}{ab} = \int\limits_0^1 \frac{dx}{ax + b\,(1-x)},$$

the integration is then performed with quantities of order μ^2/m^2 being dropped. Finally, we have, isolating expressions of the form $\delta_\mu(\hat{A}\,\hat{q} - \hat{q}\,\hat{A})$ in M,

$$\delta\mu^{\text{isob}} = \frac{e}{2m}\left(\frac{m}{\mu}\right)^2 \Bigg\{ -\frac{g_1^2}{2\pi^2}(3+5\tau_3) \tag{11}$$

$$\times\Bigg[\lambda^4\left(\frac{1}{85M_1^4} + \frac{1}{16m^3M_1}\right)$$

$$+\lambda^2\left(-\frac{m^2}{24M_1^4} + \frac{2m}{3M_1^3} + \frac{1}{3M_1^2} - \frac{1}{5mM_1}\right)$$

$$-\frac{m^4}{4M_1^4} - \frac{m^3}{3M_1^3} + \frac{m^2}{5M_1^2} + \frac{m}{3M_1} + \frac{3}{64}$$

$$+\frac{M_1}{16m} - \frac{M_1^2}{64m^2} - \left(\frac{1}{16} - \frac{m}{6M_1} + \frac{m^2}{10M_1^2}\right.$$

$$\left.+\frac{13m^3}{12M_1^3} + \frac{m^4}{3M_1^4}\right)\ln\left(1+\frac{\lambda^2}{\mu^2}\right) - \frac{2gg_1\varepsilon}{\pi^2}\tau_3\Bigg[\frac{\lambda^2}{m^2}\left(\frac{5m^2}{24M_1^2}\right.$$

$$\left.+\frac{9m}{20M_1^2} + \frac{3}{8} + \frac{3\mu s}{4M_1} + \frac{2\mu sm}{3M_1^2}\right) + \left(\frac{3m}{10M_1} + \frac{3}{64} + \frac{M_1}{16m}\right.$$

$$\left.+\frac{\mu s}{20} - \frac{\mu sm}{16M_1^2}\right)\ln\left(1+\frac{\lambda^2}{\mu^2}\right) + \frac{m^2}{2M_1^2} + \frac{7m}{9M_1} + \frac{25}{24}$$

$$+\frac{M_1}{12m} - \frac{\mu s}{32m} + \frac{7\mu s}{2M_1} + \frac{5\mu sm}{2M_1^2}\Bigg] + \frac{g_1^2}{\pi^2}\tau_3\Bigg[-\lambda^2\left(\frac{1}{4M_1^2}\right.$$

$$\left.+\frac{3}{16mM_1}\right) + \frac{5m^3}{32M_1^3} + \frac{5m^2}{3M_1^2} + \frac{m}{8M_1} + \frac{5m^4}{32M_1^4} + \frac{1}{24} +$$

$$+\frac{M_1}{24m} + \left(\frac{m^2}{18M_1^2} + \frac{m}{36M_1}\right)\ln\left(1+\frac{\lambda^2}{\mu^2}\right)\Bigg]\Bigg\}.$$

This is now to be inserted into the expression for the anomalous magnetic moment derived from the regular theory as discussed in the introduction (Eq. (2).

4. NUMERICAL RESULTS. DISCUSSION

The numerical results for $\delta\mu$ depend upon the sign of the pseudovector coupling constant g (the sign of the constant εg_1 was determined by Ritus[4]). If one employs the constant values as determined by Tamm *et al.*[3] and Ritus[4],

$$g^2 = 0.2; \quad g_1^2 = 0.13; \quad s = 2;$$

$$M_1 = m + 2.25\ \mu; \qquad \varepsilon = 1.61$$

and assumes that $g > 0$, then for $\lambda \simeq m$ we have

$$\delta\mu_p \approx 1.5\ \mu_0; \quad \delta\mu_N \approx -1.3\ \mu_0.$$

As λ increases, the absolute magnitudes of $\delta\mu_p$ and $\delta\mu_N$ increase while their ratio remains approximately one (changes slowly). Thus, with a suitably selected cut-off parameter, one can obtain sufficiently good agreement between the theory and the experimental data.

Kanazawa and Sugawara[6] obtained an approximately similar result when they also computed the effect of isobaric states on the magnetic moment of nucleons. Our calculation differs from theirs in its greater accuracy. Actually, they do not include the additional interaction of the nucleons with the electromagnetic field (coupling constant ε). Calculations indicate that this additional interaction is of the same order as other types of interactions and its inclusion is necessary. Besides this in the evaluation of the matrix elements M, the authors replaced the inverse operator $K_{\mu\nu}$ by a "nonrelativistic" approximation obtained by neglecting quantities of the type k/M_1 in comparison to 1. This is permissible in those cases if $\lambda/M_1 \ll 1$ but for satisfactory agreement between theory and experiment one must select $\lambda/M_1 \simeq 1$. Consequently, the "nonrelativistic" approximation as used by Kanazawa and Sugawara[6] is inapplicable.

In conclusion, I use this opportunity to express my gratitude to I. E. Tamm for suggesting this problem and for his continuing aid, and to Iu. A. Gol'fand, V. Ia. Fainberg and V. P. Silin for valuable discussions relating to this problem.

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Theory of Isothermal Galvanomagnetic and Thermomagnetic Effects in Semiconductors

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Isothermal galvanomagnetic and thermomagnetic effects in isotropic semiconductors are treated theoretically in the case of intermediate and strong magnetic fields.

ONE of the most effective methods of investigating the properties and parameters of semiconductors is the study of galvanomagnetic and thermomagnetic effects. A theory of these effects has been developed by a number of authors[1-7]. Most of the authors start with a quadratic dependence of the energy on the momentum, and with weakness of the magnetic field. Meanwhile, experiment has revealed many cases in which it is not legitimate to consider the effective magnetic field* φ small.

Even at room temperature, it is often necessary to deal with intermediate effective magnetic fields ($\varphi^2 \sim 1$), and sometimes even with strong ones ($\varphi^2 \gg 1$). Thus, for example, at $T = 300°$ K and $H = 10^4$ oe, $\varphi^2 \approx 1.5$ for HgSe, and $\varphi^2 = 36$ for InSb. At low temperatures we quite often deal with intermediate and strong effective magnetic fields. Davydov and Shmushkevich[3] obtained formulas for the Hall effect and for the change of electrical conductivity in the case $\varphi \gg 1$, and Madelung[6] considered the same phenomena in the case $\varphi \gtrsim 1$, but only for semiconductors with an atomic lattice.

The present work concerns the extension of the theory to the region of intermediate and strong magnetic fields, for various types of interaction of the carriers with the crystal lattice. We also determine which features of the galvanomagnetic and thermomagnetic effects depend on the statistics and on the scattering law. We consider only isothermal effects; for, as Tolpygo[5] showed, the adiabatic effects differ little in magnitude from the isothermal.

1. SEMICONDUCTORS WITH CARRIERS OF A SINGLE TYPE

Transport Equations

The kinetic equation for the distribution function $f(\mathbf{r}, \mathbf{p})$ of the carriers, in momentum ($\mathbf{p}$) and coordinate ($\mathbf{r}$) space, has in the stationary case the well-known form

$$\mathbf{v}\nabla_{\mathbf{r}} f + \mathbf{F}\nabla_{\mathbf{p}} f = -(f - f_0)/\tau. \qquad (1)$$

Here $\mathbf{v}$ is the velocity of a carrier, $\mathbf{F}$ is the external force acting on it, $f_0\left(\frac{\epsilon-\mu}{kT}\right)$ is the equilibrium distribution function, $\epsilon = \epsilon(p)$ is the energy as a function of the momentum, τ is the relaxation time, and μ is the chemical potential. In an isotropic semiconductor, the energy and the relaxation time are functions only of the magnitude p of the momentum.

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* By "effective magnetic field" we shall understand the dimensionless quantity $\varphi = uH/c$, which essentially determines the effect of the magnetic field H on the carriers of current in a semiconductor. Here u is the mobility of the carriers, and c is the speed of light.

A carrier in a magnetic field **H** and an electric field **E** is acted upon by the Lorentz force $\mathbf{F} = e(\mathbf{E} + [\mathbf{p}, \mathbf{H}]/mc)$, where e is the elementary charge, and m is the effective mass of the carrier. We shall, as usual, seek a solution of Eq. (1) in the form

$$f = f_0 + \mathbf{p}\chi(p, \mathbf{r}, T)/m. \tag{2}$$

Upon substituting **F** and f in (1), we obtain, after a few transformations, the following equation for χ:

$$\chi + \frac{e\tau}{mc}[\mathbf{H}, \chi] \tag{3}$$

$$= -\tau\left\{e\left(E - \frac{T}{e}\nabla_{\mathbf{r}}\frac{\mu}{T}\right) - \varepsilon\nabla\ln T\right\}\frac{df_0}{d\varepsilon} = \chi_0.$$

If we introduce the notation $\alpha = \mathbf{H}/H$, $\beta(p) = eH\tau(p)/mc$, the solution of (3) can be written in the form

$$\chi = \{\chi_0 + \beta[\chi_0, \alpha] + \beta^2\alpha(\chi_0, \alpha\}/(1+\beta^2). \tag{4}$$

The electric current density is

$$\mathbf{j} = e\int\frac{\mathbf{p}}{m^2}(\chi\mathbf{p})\frac{df_0}{d\varepsilon}d\mathbf{p} = \frac{4\pi}{3}e\int_0^\infty\frac{p^4}{m^2}\chi\frac{df_0}{d\varepsilon}dp. \tag{5a}$$

The density of the heat current transported by the electrons is

$$Q = \int\varepsilon\frac{\mathbf{p}}{m^2}\frac{df_0}{d\varepsilon}(\chi, p)\,d\mathbf{p} = \frac{4\pi}{3}\int_0^\infty\frac{p^4\varepsilon}{m^2}\frac{df_0}{d\varepsilon}\chi\,dp. \tag{5b}$$

Upon substituting (4) in (5a) and (5b), we get

$$\mathbf{j} = e^2J_{10}\widetilde{\mathbf{E}} - eJ_{11}\nabla\ln T \tag{6a}$$

$$+ [e^2J_{20}\widetilde{\mathbf{E}} - eJ_{21}\nabla\ln T, \alpha]$$

$$+ \alpha(e^2J_{30}\widetilde{\mathbf{E}} - eJ_{31}\nabla\ln T, \alpha);$$

$$\mathbf{Q} = eJ_{11}\widetilde{\mathbf{E}} - J_{12}\nabla\ln T \tag{6b}$$

$$+ [eJ_{21}\widetilde{\mathbf{E}} - J_{22}\nabla\ln T, \alpha]$$

$$+ \alpha(eJ_{31}\widetilde{\mathbf{E}} - J_{32}\nabla\ln T, \alpha).$$

Here

$$\widetilde{\mathbf{E}} = \mathbf{E} - (T/e)\nabla_{\mathbf{r}}(\mu/T),$$

$$J_{qr} = -\frac{4\pi}{3}\left(\frac{eH}{c}\right)^{q-1}\int_0^\infty\frac{p^4\tau^q}{m^{1+q}}\varepsilon^r\frac{df_0}{d\varepsilon}\frac{dp}{1+\beta^2}$$

$$= -\frac{4\pi}{3}\left(\frac{eH}{c}\right)^{q-1}\int_0^\infty\frac{p^3\tau^q}{m^q}\varepsilon^r\frac{df_0}{dp}\frac{dp}{1+\beta^2}$$

$(q = 1, 2, 3)$. The expansion of the integrals J_{qr} in powers of φ and $1/\varphi$ has the form:

$$J_{qr} = \frac{uN}{e}\sum_{i=0}^{i}(-1)^{l+1}a_{q+2l,r}\varphi^{q+2l-1}$$

$$\text{for } \varphi^2 \ll 1,$$

$$J_{qr} = \frac{uN}{e}\sum_{l=1}^{i}(-1)^{l-1}a_{q-2l,r}\varphi^{q-2l-1}$$

$$\text{for } \varphi^2 \gg 1.$$

$$a_{q\pm 2l,r} = (J'_{q\pm 2l,r}/J'_{00})(J'_{00}/J'_{10})^{q\pm 2l};$$

$$J'_{q\pm 2l,r} = -\frac{4\pi}{3}$$

$$\times\int_0^\infty p^3\left(\frac{\tau}{m}\right)^{q\pm 2l}\varepsilon^r\frac{df_0}{dp}dp; \quad u = eJ'_{10}/J'_{00}$$

(N is the carrier concentration). It can be shown that the following relations hold:

$$J'_{00} = -\frac{4\pi}{3}\int_0^\infty p^3\frac{df_0}{dp}dp$$

$$= 4\pi\int_0^\infty p^2 f_0\,dp = N, \quad J_{1r} + J_{3r} = J'_{1r}.$$

The expansion of the integrals J_{qr} is, in general, of asymptotic type. More explicitly, from some term onward the coefficients in the expansion increase without limit; therefore, it is necessary to break off the expansion at a term preceding the smallest term of the series.

In practice, we limit ourselves to the terms proportional to φ (for $\varphi^2 \ll 1$) and to φ^{-1} (for $\varphi^2 \gg 1$). This means that for $\varphi = 1/5$ or $\varphi = 5$, for example, the error in our formulas will not exceed 4%. Actually, the accuracy of the calculation is even greater, since in the expansion of J_{qr} in each case we neglected one term in the denominator.

Galvanomagnetic Effects

Under the isothermal condition $(\nabla T = 0)$ (6a) takes the form

$$\mathbf{j} = e^2 J_{10}\,\mathbf{E} + e^2 J_{20}\,[\mathbf{E},\alpha] + e^2 J_{30}\,(\mathbf{E}\alpha)\,\alpha. \qquad (7)$$

We consider a specimen in the shape of a rectangular parallelepiped. Let the primary electric field be directed along the x axis; let the magnetic field lie in the xz plane. Then we may set $\alpha_x = \cos\theta$, $\alpha_y = 0$, $\alpha_z = \sin\theta$, where θ is the angle between the magnetic and the primary electric fields.

The conditions that there be no electric current in the y and z directions give two equations to determine the components E_y and E_z of the electric field:

$$E_y = \frac{E_x J'_{10} J_{20} \sin\theta}{J_{10}^2 + J_{20}^2 \cos^2\theta + J_{10}J_{30}\sin^2\theta}, \qquad (8)$$

$$E_z = \frac{E_x\,(J_{20}^2 - J_{10}J_{30})\sin 2\theta}{2(J_{10}^2 + J_{20}^2\cos^2\theta + J_{10}J_{30}\sin^2\theta)}.$$

The relative change of the electrical conductivity $\sigma = j_x/E_x$ in the magnetic field can be found by substituting (8) in the expression for the x component of current:

$$\Delta\,\sigma/\sigma_0 = (J_{10}J_{30} - J_{20}^2)\sin^2\theta/(J_{10}^2 + J_{20}^2\cos^2\theta + J_{10}J_{30}\sin^2\theta), \qquad (9)$$

where $\Delta\sigma = \sigma_0 - \sigma$; σ_0 is the electrical conductivity of the semiconductor in the absence of a magnetic field. From (8) and (9) we get

$$E_z = -(\Delta\sigma/\sigma_0)E_x \operatorname{ctg}\theta. \qquad (10)$$

Thus in the general case of arbitrary statistics, scattering law and dependence of the relaxation time on the momentum, the (Hall) field E_y changes its sign upon change of sign of the magnetic field $(\theta \to \theta + \pi)$. If the magnetic field coincides in direction with the primary electric field, then the Hall field reduces to zero. The (longitudinal-transverse) field E_z does not change sign upon change of direction of the magnetic field. E_z reduces to zero in two cases: when H coincides in direction with E_x. The longitudinal-transverse effect was first treated by the authors.[8]

Since E_z and $\Delta\sigma/\sigma_0$ are mutually dependent, we shall henceforth give expressions only for $\Delta\sigma/\sigma_0$. From formulas (8) and (9) it is clear that all three of the effects considered have some anisotropy with respect to the direction of the magnetic field. On the basis of Schwarz's inequality it can be concluded that $J_{20}^2 - J_{10}J_{30} \leq 0$; consequently, $\sigma \leq \sigma_0$.

The integrals that occur in formulas (8) and (9) do not in general reduce to known functions; consequently, it does not seem possible to give a general description of the dependence of E_y and $\Delta\sigma/\sigma_0$ on the effective magnetic field, the carrier concentration, and the temperature. We therefore restrict ourselves to a consideration of limiting cases and of some special cases. To find the temperature dependence of the effects, it is necessary to assign a dependence of the distribution function, the relaxation time, and the energy upon the momentum and the temperature. We set*

$$\varepsilon(p) = p^2/2m;\; f_0(p, T) = N\,(2\pi mkT)^{-3/2} e^{-p^2/2mkT};\; \tau(p, T) = \Phi(T)\,p^{n-1}. \qquad (11)$$

Upon substituting the expansions of the integrals J_{10}, J_{20} and J_{30} in (8) and (9), we get for $\varphi \ll 1$

$$E_y = a_n \varphi E_x \sin\theta, \quad \Delta\sigma/\sigma_0 = (b_n - a_n^2)\,\varphi^2\sin^2\theta. \qquad (12)$$

When $\theta = \pi/2$, these formulas agree with the formulas derived by Tolpygo[5]. For $\varphi \gg 1$,

$$E_y = \frac{\sin\theta}{\cos^2\theta + a'_n\sin^2\theta}\,\varphi E_x; \qquad (13)$$

$$\frac{\Delta\sigma}{\sigma_0} = \frac{(a'_n - 1)\sin^2\theta}{\cos^2\theta + a'_n\sin^2\theta};$$

$$a_n = 3\sqrt{\pi}/4\Gamma(n + 3/2)/4\Gamma^2(n/2 + 2);$$

$$b_n = 9\pi\Gamma(3n/2 + 1)/16\Gamma^3(n/2 + 2),$$

$$a_n = \frac{16}{9\pi}\,\Gamma\left(3 - \frac{n}{2}\right)\Gamma\left(\frac{n}{2} + 2\right),$$

$$\Gamma(n) = \int_0^\infty x^{n-1}e^{-x}dx.$$

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* The right members of formulas (11) may be regarded as the first terms of expansions in series of appropriate functions; therefore, all the results obtained may be considered approximations for more general cases.

The mobility is

$$u = (4e/3\sqrt{\pi}m)\,\Phi(T)\,(2kT/m)^{(n-1)/2}\,\Gamma(n/2+2),$$

where the form of $\Phi(T)$ is determined by the nature of the interaction between the carriers and the crystal lattice.

From formulas (12) and (13) it is clear that the Hall field, in both limiting cases, varies linearly with the effective magnetic field; the relative change of electrical conductivity increases in proportion to φ in weak magnetic fields but approaches saturation in strong fields. In the case of strong fields, the peculiar anisotropy of both effects, shown by formulas (14), should be noted.

Even with the assumptions (11), it still does not seem possible to reduce the general expressions for the galvanomagnetic and thermomagnetic effects to known functions. This calculation can, however, be carried out for a number of special cases: (1) $n = 0$ (this corresponds to a semiconductor with an atomic lattice); (2) $n = 1$ (this corresponds to an ionic lattice at temperatures below the characteristic temperature); (3) $n = 2$ (this corresponds, according to Bloch's scheme, to a semiconductor with an ionic lattice at temperatures above the characteristic temperature).

We give the exact formulas for E_y and $\Delta\sigma/\sigma_0$ in these special cases:

1) $n=0$ (14)

$$E_y = \frac{J_2 \sin\theta}{(J_1^2+J_2^2)\cos^2\theta + J_1\sin^2\theta}\,E_x,$$

$$\frac{\Delta\sigma}{\sigma_0} = \frac{(J_1 - J_1^2 - J_2^2)\sin^2\theta}{(J_1^2+J_2^2)\cos^2\theta + J_1\sin^2\theta};$$

2) $n=1$ (15)

$$E_y = \varphi E_x \sin\theta, \quad \Delta\sigma/\sigma_0 = 0;$$

3) $n=2$ (16)

$$E_y = \frac{2\left(\frac{3\sqrt{\pi}}{4}t - t^2J_2\right)\sin\theta}{[t^4J_1^2 + (3\sqrt{\pi}\,t/4 - t^2J_2)^2]\cos^2\theta + 2t^2J_1\sin^2\theta}\,E_x,$$

$$\frac{\Delta\sigma}{\sigma_0} = \frac{[2t^2J_1 - t^4J_1^2 - (3\sqrt{\pi}\,t/4 - t^2J_2)^2]\sin^2\theta}{[t^4J_1^2 + (3\sqrt{\pi}\,t/4 - t^2J_2)^2]\cos^2\theta + 2t^2J_1\sin^2\theta}.$$

In (14)-(16) and below, the following notation is used:

$$J_1 = 1 - t^2 - t^4e^{t^2}\mathrm{Ei}(-t^2);\quad J_2 = t\,[{}^1/_2 - t^2 + \sqrt{\pi}\,t^3e^{t^2}F(t)];$$

$$F(t) = 1 - \mathrm{erf}(t);\quad \mathrm{erf}(t) = \frac{2}{\sqrt{\pi}}\int_0^t e^{-x^2}dx;\quad \mathrm{Ei}(-t) = \int_t^\infty \frac{e^{-x}}{x}dx,\ 0 < t < \infty.$$

For $n = 0$, $t = 3\sqrt{\pi}\ \varphi/4$; for $n = 2$, $t = 8/3\sqrt{\pi}\,\varphi$.

Thermomagnetic Effects

Let a temperature gradient exist along the x axis; let the magnetic field lie in the xz plane; and let the electric current in the specimen vanish. Then electric fields E_y' and E_z' appear in the directions of the y and z axes, and the thermoelectric field undergoes a certain increase E_x'. We will compute E_x', E_y' and E_z'. From (6a) under the condition $\mathbf{j} = 0$ we get a system of equations for determination of the fields E_x', E_y' and E_z'; the desired field E_x' is connected with the quantity $\widetilde{E}_x'$ by the relation

$$E_x = \widetilde{E}_x' - (T/e)\nabla_r(\mu/T) + (J_{11}'/eJ_{10}')\nabla \ln T.$$

By solving this system, we find

$$E_x' = -\frac{J_{10}(J_{10}J_{31} - J_{11}J_{30}) + J_{20}(J_{20}J_{11} - J_{21}J_{10}) + J_{20}(J_{31}J_{20} - J_{30}J_{21})}{eJ_{10}'(J_{10}^2 + J_{20}^2)}\sin^2\theta\,\frac{d\ln T}{dx},$$

$$E_y' = \frac{J_{11}J_{20} - J_{10}J_{21}}{e(J_{10}^2 + J_{20}^2)}\sin\theta\,\frac{d\ln T}{dx},\quad E_z' = -E_x'\,\mathrm{ctg}\,\theta. \qquad (17).$$

Thus the field E_y', which determines the transverse Nernst-Ettingshausen effect, changes sign with change of the sign of the magnetic field. The fields E_x' and E_z', which determine, respectively, the longitudinal Nernst-Ettingshausen effect and the longitudinal-transverse thermomagnetic effect, do not change sign with change of the sign of the magnetic field.

As the calculation shows, the anisotropy inherent in the galvanomagnetic effects does not appear in the thermomagnetic effects; in all further calculations, therefore, we will set $\theta = \pi/2$.

We consider the limiting cases of weak and strong effective magnetic fields. For $\varphi \ll 1$,

$$E_x' = (1-n)\left(b_n - \frac{a_n^2}{2}\right)\frac{k}{e}\varphi^2\frac{dT}{dx}, \tag{18}$$

$$E_y' = \frac{1-n}{2}a_n\frac{k}{e}\varphi\frac{dT}{dx}.$$

These formulas agree with those obtained in Ref. 5. For $\varphi \gg 1$,

$$E_x' = \frac{1-n}{2}\frac{k}{e}\frac{dT}{dx}, \quad E_y' = \frac{1-n}{2}a_n'\frac{k}{e}\frac{1}{\varphi}\frac{dT}{dx}. \tag{19}$$

Thus E_x' varies quadratically with the effective field φ in weak fields and approaches saturation in strong fields. E_y' increases linearly with φ in weak fields and is proportional to $1/\varphi$ in strong fields. Therefore, in the intermediate field range, i.e., for $\varphi \sim 1$, the function $E_y'(\varphi)$ must have at least one maximum.

We give the exact formulas for special cases:

1) $n = 0$

$$E_x' = -\frac{k}{e}\left\{t^2 - \frac{2J_1 + 3\sqrt{\pi}tJ_2/4}{J_1^2 + J_2^2}\right\}\frac{dT}{dx}, \tag{20}$$

$$E_y' = \frac{k}{e}\frac{2J_2 - 3\sqrt{\pi}\,tJ_1/4}{J_1^2 + J_2^2}\frac{dT}{dx};$$

2) $n = 1$ (21)

$$E_x' = E_y' = 0;$$

3) $n = 2$ (22)

$$E_x' = -\frac{k}{e}\left\{t^2 - \frac{2t^4J_1 + (15\sqrt{\pi}\,t/8)(3\sqrt{\pi}\,t/4 - t^2J_2)}{t^4J_1^2 + (3\sqrt{\pi}\,t/4 - t^2J_2)^2}\right\}\frac{dT}{dx},$$

$$E_y' = \frac{k}{e}\frac{2t^2(3\sqrt{\pi}\,t/4 - t^2J_2) - 15\sqrt{\pi}\,t^3J_1/8}{t^4J_1^2 + (3\sqrt{\pi}\,t/4 - t^2J_2)^2}\frac{dT}{dx}.$$

Formulas (20)-(22) do in fact imply the presence of a maximum in $E_y'(\varphi)$ and of a region of saturation in $E_x'(\varphi)$. Graphs of the functions $E_x'(\varphi)$ and $E_y'(\varphi)$ are given in Figs. 1 and 2.

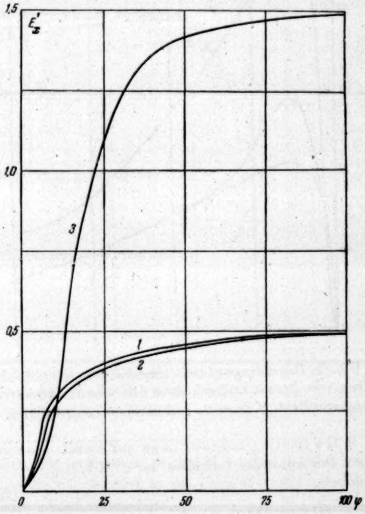

FIG. 1. Dependence of the dimensionless field of the longitudinal Nernst-Ettingshausen effect on the effective magnetic field. Curves *1, 2, 3* correspond to $n = 0, 2, 4$.

We consider the effect of a magnetic field on the electronic part of the heat conductivity. We set $E_y' = E_z' = 0$. Under isothermal conditions $\partial T/\partial y = \partial T/\partial z = 0$. Then from (6a) and (6b) we have

$$j_x = e^2 J_{10}\tilde{E}_x' - eJ_{11}\frac{d\ln T}{dx}, \tag{23}$$

$$Q_x = eJ_{11}\tilde{E}_x' - J_{21}\frac{d\ln T}{dx}.$$

From the condition $j_x = 0$ we find $E_x' = (J_{11}/eJ_{10})d \times \ln T/dx$. The coefficient of heat conductivity due to the current carriers is

$$\lambda = -\frac{Q_x}{dT/dx} = \frac{J_{10}J_{12} - J_{11}^2}{TJ_{10}}. \tag{24}$$

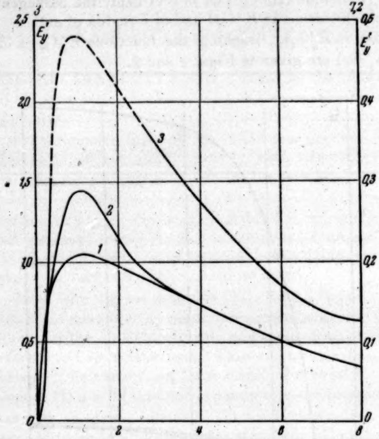

FIG. 2. Dependence of the dimensionless field of the transverse Nernst-Ettingshausen effect on the effective magnetic field. Curves *1, 2, 3* correspond to $n = 0, 2, 4$.

In the limiting cases of large and small φ we get for λ the following formulas: for $\varphi \ll 1$,

$$\lambda = (k^2T/e)\,uN\,\{2 + n/2 - b_n(n^2 - n/2 + 2)\,\varphi^2\}; \tag{25}$$

for $\varphi \gg 1$,

$$\lambda = (k^2T/e)\,uNa'_n\,(3 - n/2)\,\varphi^{-2}. \tag{26}$$

Formula (25) agrees with the corresponding formula of Ref. 5*.

From (25) and (26) it follows that the relative change of the coefficient of heat conductivity, $(\lambda_0 - \lambda)/\lambda_0$, is a quadratic function of φ in weak magnetic fields and approaches unity in strong fields.

In special cases the coefficient of the electronic part of the heat conductivity is determined by the expressions:

1) $n = 0$ (27)

$$\lambda = (k^2\,T/\varepsilon)\,uN\,[(2t^2 + 6)\,J_1 - 4]\,/\,J_1,$$

2) $n = 1$ (28)

$$\lambda = (5k^2T\,/\,2e)\,uN\,/\,(1 + \varphi^2);$$

3) $n = 2$ (29)

$$\lambda = (k^2T\,/\,e)\,uN\,[(6 + 2t^2)\,J_1 - 4]\,/\,2J_1.$$

A graph of the relative change of heat conductivity as a function of the effective magnetic field is given in Fig. 3.

From formulas (18) and (19), E'_x and $E'_y > 0$ for $n < 1$; $E'_x = E'_y = 0$ for $n = 1$; and E'_x and $E'_y < 0$ for $n > 1$. It is the absolute values of E'_x and E'_y that are plotted in the graphs 1 and 2. It is evident from Fig. 2 that the maximum value of E'_y increases with increase of n. The graphs of the function $\Delta\lambda/\lambda_0(\varphi)$ for different n's differ from one another only in the region of intermediate effective fields (Fig. 3). The graphs for the case $n = 4$, which corresponds to scattering by impurity ions and to polar conductivity at high temperatures, were drawn on the basis of the asymptotic formulas.

2. SEMICONDUCTORS WITH MIXED CONDUCTION

Galvanomagnetic and thermomagnetic phenomena in semiconductors with mixed conduction have a number of peculiarities as compared with the same phenomena in semiconductors with current carriers of a single sign. The researches of one of us[10] have shown, for example, that the appearance of a few percent of minority carriers can have an appreciable influence on the character of the thermomagnetic effects.

In the case of mixed conduction, the electric or heat current is defined as the sum of the electron current and the hole current:

$$\mathrm{j} = \mathrm{j}_+ + \mathrm{j}_-, \quad \mathrm{Q} = \mathrm{Q}_+ + \mathrm{Q}_-. \tag{30}$$

Hereafter the plus sign will denote quantities pertaining to holes, the minus sign to quantities pertaining to electrons.

The calculation will be carried out on the assumptions (11)*, except that the following expression will be assumed for f_0:

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* We remark that in Ref. 5 an error crept into formulas (34) and (34′) for the heat current: in the numerator of the fraction, $n^2 - n + 2$ should be $n^2 - n/2 + 2$. The same mistake was made by Avak'iants in Ref. 9.

* Original: (12).

$$f_0(p, T) = N(2\pi mkT)^{-3/2} \exp\left\{-\frac{1}{2kT}\left(w + \frac{p^2}{2m}\right)\right\},$$

where w is the width of the forbidden energy gap.

$m_{\mp}$ and $N_{\mp}$ are to be interpreted as the effective mass and concentration of the electrons or holes. It is also assumed that the scattering mechanism is the same for electrons and for holes, i.e., that $n_- = n_+ = n$.

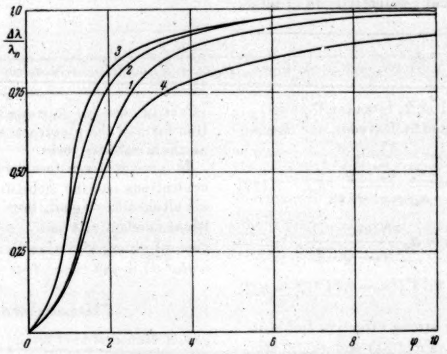

FIG. 3. Dependence of the relative change of the electronic part of the heat conductivity on the effective magnetic field. Curves *1, 2, 3, 4* correspond to $n = 0, 1, 2, 4$.

Galvanomagnetic Effects

The expression for the total electron and hole current retains the form (7), but the coefficient J_{qr} must be redefined as follows:

$$J_{10} = J_{10}^{+} + J_{10}^{-},\quad J_{20} = J_{20}^{+} - J_{20}^{-},\quad J_{30} = J_{30}^{+} + J_{30}^{-}. \tag{31}$$

With the interpretation (31), formulas (8) and (10) may be carried over to the case of semiconductors with mixed conduction.

We will obtain expressions for the Hall field and for the relative change of electrical conductivity in the limiting cases of weak and strong effective magnetic fields. For $\varphi_{\pm} \ll 1$,

$$E_y = a_n \frac{N_+u_+\varphi_+ - N_-u_-\varphi_-}{N_+u_+ + N_-u_-} E_x \sin\theta, \tag{32}$$

$$\frac{\sigma_0 - \sigma}{\sigma_0} = \frac{b_n(N_+u_+ + N_-u_-)(N_+u_+\varphi_+^2 + N_-u_-\varphi_-^2) - a_n^2(N_+u_+\varphi_+ - N_-u_-\varphi_-)^2}{(N_+u_+ + N_-u_-)^2} \sin^2\theta. \tag{33}$$

For weak effective fields the Hall field, as in the case of semiconductors with carriers of one sign, depends linearly on $\varphi_{\pm}$; but it may change sign in its variation with the concentration and mobility ratios of the electrons and holes. The relative change of electrical conductivity is proportional to $\varphi_{\pm}^2$, as in the case of semiconductors with carriers of one sign; but for $n = 1$ it does not reduce to zero.

For $\varphi_{\pm} \gg 1$ we consider two cases:

$$\text{(a) } N_+ \neq N_-:\quad E_y = \frac{(N_+ - N_-)(N_+\varphi_+ + N_-\varphi_-)}{(N_+ - N_-)^2\cos^2\theta + a_n'(N_+/u_+ + N_-/u_-)(N_+u_+ + N_-u_-)\sin^2\theta} E_x \sin\theta. \tag{34}$$

$$\frac{\sigma_0-\sigma}{\sigma_0}=\frac{a_n'(N_+/u_+ + N_-/u_-)(N_+n_+ + N_-u_-)-(N_+-N_-)^2}{(N_+-N_-)^2\cos^2\theta + a_n'(N_+/u_+ + N_-/u_-)\sin^2\theta\,(N_+u_+ + N_-u_-)}\sin^2\theta. \tag{35}$$

In the case of strong effective fields, the expression for the electrical conductivity is of interest:

$$\sigma=\sigma_0\frac{(N_+-N_-)^2}{(N_+-N_-)^2\cos^2\theta + a_n'(N_+/u_+ + N_-/u_-)(N_+u_+ + N_-u_-)\sin^2\theta}. \tag{36}$$

For $\theta=\pi/2$ and $n=0$, 1 or 2, formulas (36) reduces to formulas obtained by Davydov and Shmushkevich[3].

$$\text{(b) } N_+=N_-=N: \qquad E_y=\frac{c_n}{a_n'}\frac{\varphi_+-\varphi_-}{\varphi_+\varphi_-\sin\theta}E_x, \tag{37}$$

$$\frac{\sigma_0-\sigma}{\sigma_0}=1,\ \sigma=a_n'\frac{eN(u_++u_-)}{\varphi_+\varphi_-\sin^2\theta},$$

$$c_n=(4/3\sqrt{\pi})^3\,\Gamma(7/2-n)\,\Gamma^2(2+n/2).$$

From the formulas for strong effective fields it follows that the functions $E_y(\varphi_\pm)$ and $\sigma(\varphi_\pm)$ are different for different ratios between the concentrations of the electrons and of the holes. In the case $N_+\neq N_-$, $E_y(\varphi_\pm)$ increases with increase of the effective field; $\sigma(\varphi_\pm)$ approaches saturation. For $N_+=N_-$, $E_y(\varphi_\pm)$ and $\sigma(\varphi_\pm)$ decrease with increase of the effective field. Formulas (37) for E_y and σ are correct if θ differs considerably from zero. If θ is small, it is necessary in deriving the formulas to take account of terms of fourth order in $1/\varphi_\pm$. Then the functions $E_y(\varphi_\pm)$ and $\sigma(\varphi_\pm)$ will have the same character as in the case $N_+\neq N_-$. Formulas (37) imply a very unusual type of dedendence of the effects on the angle between the magnetic field and the primary electric field: upon decrease of the angle θ (provided θ differs considerably from zero), the Hall field and the electrical conductivity increase. The sign of the Hall effect in case (a) depends mainly on the concentration ratio of the electrons and holes; in case (b), on their mobility ratio.

In a conductor with mixed conduction, if the concentrations and the mobilities of the carriers are simultaneously equal, then $E_y=0$, and the electrical conductivity is $\sigma=a_n'\sigma_0/\varphi^2\sin^2\theta$ in the case of strong effective fields and $\sigma=\sigma_0(1-b_n\varphi^2\times\sin^2\theta)$ in the case of weak fields.

Thermomagnetic Effects

If in formulas (17) for E_x' and E_y' we make the substitutions

$$J_{11}={}^1/_2(J_{10}^+-J_{10}^-)w+J_{11}^+-J_{11}^-,$$

$$J_{21}={}^1/_2(J_{20}^++J_{20}^-)w+J_{21}^++J_{21}^-,$$

$$J_{31}={}^1/_2(J_{30}^+-J_{30}^-)w+J_{31}^+-J_{31}^-,$$

and also use (31), we get the general expressions for the longitudinal and transverse Nernst-Ettingshausen effects in semiconductors with mixed conduction. The relation between E_z' and E_x' stays the same even for semiconductors with mixed conduction. Therefore, only the expressions for E_x' and E_y' will be considered below. We consider the limiting cases of weak and strong effective fields, on the supposition that $\theta=\pi/2$:

For $\varphi_\pm\ll 1$,

$$\begin{aligned}E_x'=\frac{k}{e}\frac{1}{(N_+u_++N_-u_-)^3}\Big\{\Big\{&(1-n)\Big\{\Big(b_n-\frac{a_n^2}{2}\Big)(N_+^3u_+^3\varphi_+^2-N_-^3u_-^3\varphi_-^2)\\&+N_+N_-u_+u_-\Big[b_n(N_+u_+\varphi_+^2-N_-u_-\varphi_-^2)-\frac{a_n^2}{2}(N_-u_--N_+u_+)\varphi_+\varphi_-\Big]\Big\}\\&-N_+N_-u_+u_-\Big\{\Big[(4+2n)b_n-\frac{3n+8}{2}a_n^2\Big](N_+u_+\varphi_+^2-N_-u_-\varphi_-^2)+(4+2n)b_n(N_+u_+\varphi_+^2\\&-N_-u_-\varphi_-^2)-\frac{3n+8}{2}a_n^2(N_-u_--N_+u_+)\varphi_+\varphi_-\Big\}\frac{w}{kT}\Big\}\Big\}\frac{dT}{dx};\end{aligned} \tag{38}$$

$$E'_y = a_n \frac{k}{2e} \frac{1}{(N_+u_+ + N_-u_-)^2} \Big\{ (1-n)(N_+^2 u_+^2 \varphi_+ + N_-^2 u_-^2 \varphi_-) - (3n+7) N_+ N_- u_+ u_- (\varphi_+ + \varphi_-) \left(1 + \frac{2}{3n+7} \frac{w}{kT}\right) \Big\} \frac{dT}{dx}; \tag{39}$$

For $\varphi_\pm \gg 1$: (a) $N_+ \neq N_-$:

$$E'_x = \frac{k}{2e} \frac{(1-n)(N_+^2 u_+ + N_-^2 u_-) + (n+9) N_+ N_- (u_+ + u_-) \left(1 + \frac{2}{n+9} \frac{w}{kT}\right)}{(N_+ - N_-)(N_+u_+ + N_-u_-)} \frac{dT}{dx},$$

$$E'_y = a'_n \frac{k}{2e} \frac{1}{(N_+ - N_-)^2 \varphi_+ \varphi_-} \Big\{ (1-n)(N_+^2 \varphi_- + N_-^2 \varphi_+) - (11-n) N_+ N_- (\varphi_+ + \varphi_-) \left(1 + \frac{2}{11-n} \frac{w}{kT}\right) \Big\} \frac{dT}{dx}; \tag{40}$$

(b) $N_+ = N_- = N$:

$$E'_x = \frac{5k}{e} \frac{(c_n - a_n'^2)(u_+ - u_-)}{u_+ + u_-} \left(1 + \frac{w}{5kT}\right) \frac{dT}{dx}, \tag{41}$$

$$E'_y = -\frac{5k}{e} \frac{\varphi_+ \varphi_-}{a'_n (\varphi_+ + \varphi_-)} \left(1 + \frac{w}{5kT}\right) \frac{dT}{dx}.$$

The formulas for the thermomagnetic effects show that for $\varphi_\pm \ll 1$, the longitudinal and transverse Nernst-Ettingshausen fields depend on the effective field in the same way as in semiconductors with carriers of one sign. In the case of strong effective fields the dependence of E'_x and E'_y on $\varphi_\pm$ is different for different ratios between the concentrations of the electrons and of the holes. When $N_+ \neq N_-$, the dependence remains of the same type as in semiconductors with carriers of one sign. If $N_+ = N_-$, however, the transverse Nernst-Ettingshausen field does not decrease, but increases, with increase of the effective field; the longitudinal Nernst-Ettingshausen field approaches saturation but the saturation value is different from the saturation value for $N_+ \neq N_-$.

Estimation indicates that formulas (37) and (41) are applicable only when the electron and hole concentrations are quite close to each other in magnitude. Thus if we put $\varphi = 5 \gg 1$(by φ must be understood the larger of the quantities φ_+ and φ_-), the indicated formulas may be used when the concentration ratio is not less than 0.96.

Finally, we give formulas for the electronic part of the heat conductivity of semiconductors in the limiting cases of weak and strong effective fields. In view of the extreme unwieldiness of the general formulas for the case $\varphi_+ \ll 1$, we give a formula obtained on the supposition that $w/kT \gg 1$, by neglecting terms not containing w/kT or proportional to it, and keeping only terms proportional to $(w/kT)^2$:

$$\lambda \approx \frac{w^2}{eT} \frac{N_+N_-u_+u_-}{N_+u_+ + N_-u_-} \left\{ 1 - a_n \frac{N_+u_+\varphi_-^2 + N_-u_-\varphi_+^2}{N_+u_+ + N_-u_-} \right\}, \tag{42}$$

$$\lambda_0 = \frac{k^{+}T}{e} \left\{ \left(2 + \frac{n}{2}\right)(N_+u_+ + N_-u_-) + (4+n)^2 \frac{N_+N_-u_+u_-}{N_+k_+ + N_-u_-} \left(1 + \frac{1}{4+n} \frac{w}{kT}\right)^2 \right\}, \tag{43}$$

λ_0 is the electronic part of the heat conductivity in the absence of a magnetic field. For $\varphi_+ \gg 1$,

$$\lambda = a'_n \frac{k^2T}{e} \left\{ \left(3 - \frac{n}{2}\right) \left(\frac{N_+u_+}{\varphi_+^2} + \frac{N_-u_-}{\varphi_-^2}\right) + \frac{N_+N_-u_+u_-(6-n)^2}{N_+u_+\varphi_+^2 + N_-u_-\varphi_-^2} \left(1 + \frac{1}{6-n} \frac{w}{kT}\right)^2 \right\}. \tag{44}$$

Thus different ratios between the electron and hole concentrations, in the case of strong fields, do not influence in an essential way the dependence of the electronic part of the heat conductivity on the effective field, as was true of the Nernst-Ettingshausen fields.

We consider it a welcome duty to express our thanks to Kh. I. Amirkhanov and V. P. Zhuze for their interest in the work and for discussion of its results.

Note added in proof: Recently one of the authors (F. G. Bass) has succeeded in showing that on the assumption of an arbitrary isotropic scattering law, an arbitrary form of the collision integral, and any statistics, the dependence of properties on magnetic field in the galvanomagnetic and thermomagnetic effects, in the limiting cases $\varphi^2 \gg 1$ and $\varphi^2 \ll 1$, is the same as that obtained in the present article under the assumptions of formulas (11).

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Letters to the Editor

On a Rational System of Symbols for Fundamental Particles

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ACCORDING to the latest investigations* on the systematics of the elementary interactions, there are four fundamental parameters. These parameters E, n, s and ν determine the type of the process for the formation and decay of the elementary particles, as well as the place of the particles in the natural systematics shown in Table I**. Here $E = Q/e$ is the difference between the positive and negative charges; n is the difference between the number of heavy particles (barions) and heavy antiparticles (antibarions), or the "nuclear charge"***[4-6]; s is the "strangeness"[1] or v is the charge[7]; ν is the difference between the number of light particles (leptons) and light antiparticles (antileptons), or the "neutrinic charge"***[8,9].

The number n, as well as E, is conserved in all the known reactions, and its conservation implies a certain absolute law on the conservation of the number of barions. The number s is conserved only in strong electromagnetic interactions, and can change by a unit in the weak interactions of the slow decay of a hyperon or of a heavy meson[1,7]. Instead of s, one can use some other number, subject to the same rules: the "neutronic charge" ϵ[2,3], the "attribute" a[11] or the projection μ_3 of the μ-isotopic spin[12].

We have shown[2] that the numbers s, ϵ, a, μ_3 and the projection τ_3 of the usual isotopic spin have the following relationship:

$$\varepsilon + E = s + n = -a = 2\mu_3; \tag{1}$$

$$E = \mu_3 + \tau_3, \quad \varepsilon = \mu_3 - \tau_3;$$

$$\tau_3 = {}^1/_2\,(E - \varepsilon), \quad \mu = {}^1/_2\,(E + \varepsilon).$$

It has also been shown[2] that the quantity ϵ can indeed be considered as analogous to the electric charge; this is why we think it is rational to use this quantity rather than s, a or μ_3.

The conservation of the number ν expresses the conservation of the number of leptons, in analogy with the fact that the conservation of N expresses the conservation of the number of barions. It is possible (as assumed in Ref. 9), that the conservation of the number ν is also an absolute conservation law. This point is also discussed in Ref. 13.

It is worthy noting that the names "nuclear charge" for the quantity n and "neutrinic charge" for the quantity ν can be justified only by the fact that some integers are conserved, in analogy to the conservation of the electric charge. However, there are so far no bases to believe that the nuclear and the neutronic "charges" create any fields in analogy to the electric charge. Therefore, the term "charge" for the quantities n and ν should be used purely in a conventional sense. The best way to name the quantities E, ϵ, n and ν is therefore: E = electric number, ν = neutronic number, n = barionic number, $\lambda = -\nu$ is the leptonic number.

The numbers E, ϵ, n and λ determine the place of the particle in the schematics, its basic properties, and its allowed formation and decay reactions; it is therefore rational to introduce these numbers into the symbols describing the elementary particles. We propose to write the numbers E, ϵ, n and λ as indices of the symbol Z of the particles, according to the scheme

$${}^{\lambda}_{n}Z^{E}_{\varepsilon} \tag{2}$$

When a reaction is described, the indices (λ, n) on the left-hand side could be confused with the indices on the right-hand side. To avoid this, we will denote the latter ones by the signs + and −, and the former ones by the signs 1 and − 1. In the case where any of the numbers is equal to zero, the corresponding place will be left blank. If we chose for the barions and for the mesons the numbers E, ϵ and n as they should according to Refs. 1 and 2, and chose for the leptons the number according to Ref. 9 ****, we get Table II instead of Table I.

The symbolism of Table II presents several advantages over that of Table I. For instance, to write all the possible reactions for the formation of particles, satisfying the condition $\Delta\epsilon = 0$ (or $\Delta s = 0$) it is sufficient to require that the sum of the

TABLE I

γ	
ν e^-	$\tilde{\nu}$ e^+
μ^+	μ^-
$\pi^+\ \pi^0\ \pi^-$	
$k^+\ k^-$ $k^0\ \tilde{k}^0$	
p n	$\tilde{p}$ $\tilde{n}$
Λ^0 $\Sigma^+\ \Sigma^0\ \Sigma^-$	$\tilde{\Lambda}^0$ $\tilde{\Sigma}^-\ \tilde{\Sigma}^0\ \tilde{\Sigma}^+$
Ξ^- Ξ^0	$\tilde{\Xi}^+$ $\tilde{\Xi}^0$

TABLE II

γ	
$^{-1}\nu$ $^{-1}e^-$	$^{1}\nu$ $^{1}e^+$
$^{-1}\mu^+$	$^{1}\mu^-$
$\pi^+_-\ \ \pi\ \ \pi^-_+$	
$k^+\ k^-$ $k_+\ k_-$	
$_1p^+$ $_1n_+$	$_{-1}p^-$ $_{-1}n_-$
$_1\Lambda$ $_1\Sigma^+_-\ _1\Sigma\ _1\Sigma^-_+$	$_{-1}\Lambda$ $_{-1}\Sigma_+\ _{-1}\Sigma\ _{-1}\Sigma^+_-$
$_1\Xi^-$ $_1\Xi_-$	$_{-1}\Xi^+$ $_{-1}\Xi_+$

indices of any type, on the right or on the left, be identical; e.g.,

$$_1p^+ + {_1p^+} \rightarrow \Lambda + k^+ + {_1p^+},$$

$$_1p^+ + {_1p^+} \rightarrow {_1\Sigma^{+}_{-}} + k_+ + {_1p^+};$$

$$\pi^{-}_{+}\, {_1p^+} \rightarrow {_1\Lambda} + k_+,$$

$$\pi^{-}_{+} + {_1n_+} \rightarrow {_1\Xi^{-}} + k_+ + k_+.$$

and it is obvious that the reaction

$$_1n_+ + {_1n_+} \rightarrow {_1\Lambda} + {_1\Lambda}$$

is forbidden, because $\Delta\epsilon = -2$.

In the case of the slow decays of hyperons or mesons, in which $\Delta\epsilon = \pm 1$, we have, for instance,

$$_1\Xi^{-} \rightarrow {_1\Lambda} + \pi^{-}_{+},\ {_1\Xi_{-}} \rightarrow {_1\Lambda} + \pi,$$

$$_1\Sigma^{+}_{-} \rightarrow {_1p^+} + \pi,\ {_1\Sigma^{-}_{+}} \rightarrow {_1n_+} + \pi^{-}_{+},$$

$$_1\Sigma^{+}_{-} \rightarrow {_1n_+} + \pi^{+}_{-},$$

$$_1\Lambda \rightarrow {_1p^+} + \pi^{-}_{+},\ {_1\Lambda} \rightarrow {_1n_+} + \pi,$$

$$k^+ \rightarrow \pi^{+}_{-} + \pi,\ k_+ \rightarrow \pi^{+}_{-} + \pi^{-}_{+},$$

$$\pi^{+}_{-} \rightarrow {^{-1}\mu} + {^{1}\nu},$$

$$_1n^+ \rightarrow {_1p^+} + {^{1}e^-} + {^{-1}\nu}.$$

It is easy to see that the not yet discovered particles could be the following: 1) heavy mesons M^{+}_{+} and M^{-}_{-} (these particles were predicted[1,2,10] or identified with the τ-mesons[12]); 2) barions or antibarions B^{+}_{+}, $_1B^{-}_{-}$, $_{-1}B^{-}_{-}$, $_{-1}B^{+}_{+}$ (these particles were predicted[1]); 3) leptons (analogous to all the possible barions).

All these supposedly possible particles have the quantum numbers equal to -1, 0 or $+1$ as the discovered particles.

We consider that the proposed symbolism reflects better than any other the main features of the natural system.

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* See, e.g., Refs. 1 and 2.

**In this Table all the heavy mesons are denoted by a single symbol k. It is assumed (as in Ref. 1) that the θ-and τ-mesons differ only by their spin and their parity, but have the same numbers E and s. This Table also does not allow for the possibility of the existence of two types of π^0-mesons[3].

*** In our previous papers[2,3,10], the terms "nuclear" or "neutrinic" charges were used for the numbers $N = -12$. However, the term "nuclear charge" is usually reserved to the number n.

**** In Ref. 9, the value $\epsilon = 0$ is assigned to all the discovered leptons; this is not the only possibility.

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Remarks on Bilocal Field Theory

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V. A. ZHIROV has opposed three arguments against my bilocal field theory: 1) in the early stages of the theory, the formulas are not compatible with themselves; 2) in a later stage of the formulation, the equations are compatible but admit some physically inadmissible solutions, with a space-like energy-momentum 4-vector; 3) in the theory, the question of the introduction of interaction remains open.

In answer to the first argument, I should like to note that the bilocal equations of my theory are always compatible with themselves, and that the apparent "incompatibility" can rise only in the automatic applying of my formulas without consideration of their limitations, or in the confusion among the notations for the field quantities in six-dimensional or eight-dimensional multiplicites. I admit to be guilty of not having sufficiently clarified these points.

The bilocal field quantities are two types of functions of the variables χ_μ and r_μ, where χ_μ denotes the center of the particles and r_μ the intrinsic degrees of freedom. A Fourier transformation enables us to go from the χ_μ to the p_μ variables, the latter being the components of the energy-momentum 4-vector.

The bilocal varieties and the bilocal fields are limited by the following conditions:

$$p_\mu r_\mu = 0,\ (\mathrm{I});\ \ p_\mu^2 + r_\mu^2 = 0\ \ (\mathrm{II}) \qquad (1)$$

(we use a system of units where $c = h = l = 1$, where l is the fundamental length; therefore, all the magnitudes are dimensionless).

One has to differentiate between the field quantities φ, which depend only on 6 independent variables, and the quantities ψ, defined on an eight-dimensional multiplicity. φ and χ are related by

$$\psi(p, r) = N\delta(p_\mu r_\mu)\,\delta(p_\mu^2 + r_\mu^2)\,\varphi(p, r), \qquad (2)$$

where N is the normalizing factor. ψ satisfies the following supplementary condition:

$$p_\mu r_\mu \psi = 0,\ (\mathrm{I});\ (p_\mu^2 + r_\mu^2)\,\psi = 0;\ (\mathrm{II}). \qquad (3)$$

Moreover, ψ satisfies an equation which is a generalization of the Schrödinger-Fock-Gordon equation

$$(p_\mu^2 + k_\mu^2)\,\psi = R\psi, \qquad (4)$$

where $k_\mu = -i\partial/\partial r_\mu$ and R is an operator taking into account the strength of the reactions which arise as a result of the limitations (1) or (3). Furthermore, one can raise the question about the equations and the supplementary conditions satisfied by φ. The condition (1) means that, if p_μ is time-like, r_μ is space-like, and vice versa. For space-like p_μ, φ is limited by the conditions

$$p_\mu k_\mu \varphi = 0, \qquad (\mathrm{I})$$

$$\varphi = \text{zeroth order single-valued function of } r_\mu. \qquad (\mathrm{II})$$

Indeed, the condition (I) gives the result that φ does not depend on r_4 in the rest system $(\mathbf{p} = 0)$. Using condition (II) one then sees that φ does not depend on $|\mathbf{r}|$, i.e., φ is a function of the angles φ and θ only, in agreement with condition (1). On the other hand, in the case of a space-like p_μ, the vector r_μ is time-like and the relations (I) and (II) have to be replaced by: $(r_\mu \partial/\partial p_\mu)\,\varphi = 0$ and φ = a zeroth order single-valued function of p_μ. The term R in Eq. (4) has to be properly replaced. Therefore, by choosing the conditions (I) and (II) we are bound to choose time-like p_μ's.

In addition to (I) and (II) we have to introduce another equation, analogous to (4). For time-like p_μ this will simply be

$$(p_\mu^2 + k_\mu^2)\,\varphi = 0, \qquad (5)$$

which differs from (4) by the absence of the R-term. Eq. (5) is obviously compatible with (I) and (II) (but it is not compatible with the conditions chosen in the case of space-like p_μ's).

Using condition (I), we see that, in the rest system $(\mathbf{p} = 0)$, the term $k_\mu^2 \varphi$ is reduced to $\mathbf{k}^2 \varphi$; using condition (II) we see that this term becomes $n(n+1)r^{-2}\varphi$. Finally, applying the second of the conditions (1), we obtain the mass spectrum

$$m_n = [n(n+1)]^{1/4}. \qquad (6)$$

The eigenfunctions φ_n of this spectrum are related to the local Fiertz field by

$$\varphi_n(p, r) = \varphi(p)_{\mu_1\mu_2\cdots\mu_n}\, r_{\mu_1} r_{\mu_2} \cdots r_{\mu_n} / (r_\nu^2)^{n/2}. \qquad (7)$$

In this sense, the bilocal quantities are the generating functions of the local Fiertz fields. The latter form the whole irreducible unitary representation of the Lorentz group, and there are, therefore, no other solutions which would be mathematically possible but physically impossible. All other solutions either do not satisfy the supplementary conditions or (for space-like p_μ) are not related to the irreducible unitary representation of the Lorentz group. The solutions with time-like p_μ form a closed system and include all the Hilbert space.

Let us finally say a few words about the problem of the interaction. This problem is so far absolutely unclarified, and the usual requirements applied in the local theory are not applicable when the bilocal interaction is introduced. The same considerations allow us to hope that a criterion for the interaction will be found in the future, which will considerably differ from the (unmistakably wrong) criterion used now.

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Effect of Multiple Thermal Ionization on the Specific Heat of Gases

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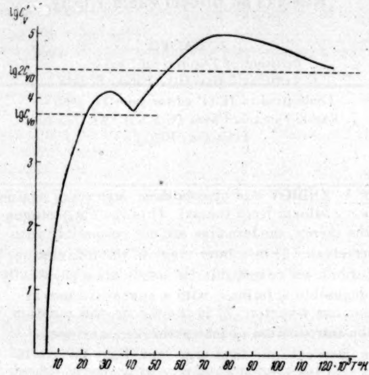
ONE of us[1] has calculated the specific heat of gases at high temperatures, taking into account a single thermal ionization. The purpose of this note is to consider the effect of multiple thermal ionization on the specific heat of gases.

We follow the method used in Ref. 1. The result is that the specific heat of the gas at high temperatures can be calculated from the formula

$$C_V = C_{V0} + C'_V \quad (1)$$



where $C_{V0} = 3/2\, nk$ is the specific heat of the non-ionized gas at constant volume; C'_V is the correction to the specific heat, due to the thermal ionization, with $C'_V = \sum_m C_{Vm}$; in the latter formula, C_{Vm} is the correction to the specific heat, due to the thermal ionization of the gas, with a $(m-1)$-fold ionization; C_{Vm} is determined by the following expression:

$$\begin{aligned} C_{Vm} = {} & (k / 4 f (kT)^2 L) \{[3 kT (2 J_m - J_{m-1}) \\ & + 2 J_m (J_m - J_{m-1})] [nf (m+1) - L + 1] \\ & + {}^{15}/_2 (kT)^2 + {}^{15}/_2 (kT)^2 (1 - L) \\ & + {}^{21}/_2 nf (m+1) (kT)^2 + 3 nf (kT)^2 (m-1) L\}; \end{aligned} \quad (2)$$

$$f = (g_{m-1} / 2 g_m) (2\pi / m')^{3/2} h^3 (kT)^{-3/2} \exp (J_m / kT),$$

$$L = [1 + n^2 f^2 (m^2 + 1 - 2m) + 2 nf (m+1)]^{1/2}.$$

It follows from (2) that the specific heat of gases at high temperature has a strong dependence on the temperature. To give an example, we have calculated the effect on the atomic oxygen gas; the correction C'_V is plotted on a logarithmic scale, vs. the temperature, in the range where only single and double ionizations are effective (solid line). One sees from the graph that the dependence of the specific heat on the temperature is represented by a curve showing steps and maxima in the temperature interval where a particular ionization order prevails. These maxima are several times higher than the specific heat of the non-ionized gas. For comparison, the specific heat of the non-ionized gas has been plotted on the same graph, as a dotted line.

The dependence of the specific heat on the temperature at higher temperatures will have the same character.

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Elastic (p–p)–Scattering and Peculiarities of Interaction between Pions and Nucleons

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STRONG interaction between a pion and a nucleon, which exerts a significant effect on the process of pion production in the collision of a nucleon with a nucleon, may develop also in the process of the elastic (p–p)–scattering. Two mechanisms are possible. One of these can be schematically expressed in the form

$$p + p \to (\pi + N + p)^* \to p' + p' \qquad (1)$$

and it corresponds to the process of resonance scattering. The second is inevitably produced as a result of the known optical relationship between elastic and inelastic processes. In so far as the values of the elastic and inelastic cross sections owing to this relationship are of the same order, and the cross section of the inelastic processes

$$N + N \to \pi + N + N' \qquad (2)$$

is comparatively large at energies beginning with 499 mev, therefore the role of the second factor may be considerable.

The calculation of the probability of the elastic (p–p)–scattering taking into account the virtual mechanism (1), carried out by Austern[1], gives values for the differential cross section differing by two orders from the values observed in experiment. This permits us to draw the initial conclusion that the peculiarities in the elastic (p–p)–scattering, apparently, are completely dependent on the second mechanism.

A juxtaposition of the elastic (p–p)–scattering with the processes of interaction between a pion and a nucleon can be made when these processes are compared at equal values of total energy in a center-of-mass system of the colliding particles. Moreover, it is also necessary to take into account the energy corresponding to the rest mass of the pion. For the value characterizing the probability of scattering at a given angle, it is necessary to consider the derivative

$$k^2 \, d\sigma(\theta) / d\omega = \alpha(\theta), \qquad (3)$$

where k is the wave vector of the colliding particles in a center-of-mass system, and $d\sigma/d\omega$ is the differential cross section of scattering at the angle θ in a c. m. system.

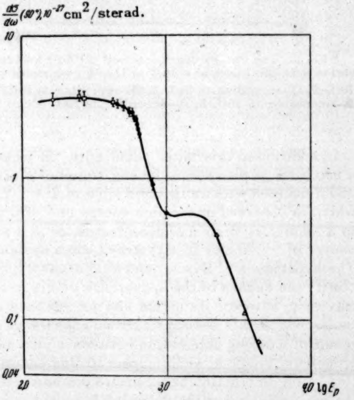

FIG. 1. The energy dependence of the elastic (p–p)–scattering at the angle 90°. ☐ –according to Ref. 2, O–according to Ref. 3, ●–according to Ref. 4, ▲–according to Ref. 5, Δ–according to Ref. 6.

Figure 1 gives the results of measurements of the differential cross section of the elastic (p–p)–scattering at the angle of 90° in the energy region from 160 mev to 4.4 bev taken from the investigations.[2-6] Figure 2 gives the dependence of the value α (90°) on the energy of the incident proton taken from the data of the above-mentioned investigations. As is evident from this figure, the latter curve (90°) has a maximum at total energy in a c. m. system at approximately 280 mev. It is precisely at these values of total energy in a c. m. system that the known peculiarities of the process $p + p \to \pi^+ + d$ are observed, as well as of the simpler processes $\gamma + p \to \pi^+ + p$ and $\pi^+ + p \to \pi^+ + p$. The appearance of this maximum is generally associated with the peculiarities of the interaction between a pion and a nucleon in states with an isotopic spin of $T = 3/2$.

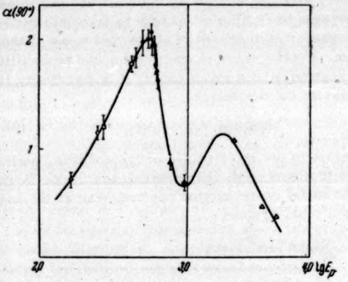

FIG. 2. The energy dependence of α (90°) for the states with the isotopic spin $T = 1\frac{1}{2}$ ☐ –according to Ref. 2, O–according to Ref .3, ●–according to Ref. 4, ▲–according to Ref. 5, Δ–according to Ref. 6.

In addition to this latter peculiarity, at present a maximum is also known for the states of a pion and and a nucleon with an isotopic spin of $T = 1/2$, which is observed at the pion energy of ~ 800 mev in a laboratory system of coordinates, or at a total energy of ~ 600 mev in a center-of-mass system. The hypotheses of Dyson[7] and of Brueckner[8] clarify the causes of the appearance of this second maximum, however its nature has not yet been established, If this second maximum appears as a result of a strong interaction between a pion and a nucleon, then one would expect to find peculiarities in the interaction between two protons in the same energy region. Initial data on the value α (90°) at energies higher than one bev indicate the presence of this predicted peculiarity, namely: as is evident from Fig. 2, in the energy region (1.5 – 2) bev the value α (90°) undergoes nonmonotonic change. The total energy in a center-of-mass system corresponding to this region is equal to 600–700 mev.

The states of a two nucleon system with $T = 0$ cannot experience the effect of strong interaction between a pion and a nucleon in the states with $T = 3/2$, since $T_{\pi N} = 3/2$ and $T_N = 1/2$ can produce only the states with $T = 2$ and $T = 1$. Therefore, the dependence curves of α (90°) for the states of two nucleons with $T = 0$ should not possess any of the peculiarities which can be observed for $T = 1$. Although the experimental data substantiate this prediction, they do not permit us to draw such a conclusion with complete certainty.

The maximum of the dependence of the total cross section of the interaction between a pion and a nucleon in the states with $t = 1/2$ at total energy in a center-of-mass system of ~ 600 mev should appear in the processes of the elastic scattering of a nucleon on a nucleon in the states with $T = 1$ and $T = 0$. This permits us to draw the conclusion that the dependence curve of α (90°) for the states of two nucleons with $T = 0$, beginning with the energy (0.8–1.0) bev, should reproduce almost completely the corresponding curve for the states of two nucleons with $T = 1$.

The author expresses his gratitude to L. I. Lapidus for valuable comments and discussion of the problem.

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Interaction between Negative Pions and Helium Nuclei at 330 mev Energy

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AT present there exist few investigations devoted to the interaction between pions and nuclei at energies higher than 200 mev. In particular, very little data have been obtained on the interaction between pions and the simplest nuclei. Below we will describe the results of the study of the interaction between negative pion and helium nuclei at 330 mev energy.

For the observation of simple interactions we have used a diffusion chamber of 270 mm in diameter which was filled with helium at a pressure of

14 atm. The results are based on an analysis of 97 events of interaction obtained from an examination of approximately 13,000 photographs.

In the collisions of fast mesons with α-particles the following main processes are possible:

$$\pi^- + \mathrm{He}^4 \rightarrow \begin{cases} \pi^- + \mathrm{He}^4 - \text{elastic scattering} & (1)\\ \pi^- + (2p,\ 2n) - \text{inelastic scattering with no change in charge} & (2)\\ \pi^0 + (1p,\ 3n) - \text{scattering with change of charge} & (3) \end{cases}$$

$$(1p,\ 3n) - \text{nonradiative capture} \qquad (4)$$

The nucleons in these reactions may be emitted as deuterons, tritons, He^3 nuclei and free nucleons. At a given energy of pions processes of meson production are also possible.

In the reactions enumerated the events of elastic scattering at angles larger than 10° can be easily identified from the general kinematic conditions. The identification of elastic scattering at smaller angles presents considerable difficulties, since the resulting fissioning nuclei (α-particles) have a very short path. It is possible, however, with a sufficient degree of certainty, to assume for the events of elastic scattering in the region of small angles any sharp deviation of the particles from its initial direction, since the overwhelming majority of inelastic processes, capable of serving as a main source of errors, is accompanied by visible tracks of the products of splitting. The lower limit of the registration angle of the elastic scattering is given by the maximum angle of $(\pi - \mu)$–disintegration equal to 5.1° at 330 mev energy.

The events of inelastic scattering with no change in charge were identified by the presence among the reaction products of a single track with ionization approximating the minimum. We did not succeed in identifying individually any events of exchanged interaction [reactions (3) and (4)]. The exception was the absorption of pions by the $(p\text{–}p)$–pairs, when as a result of interaction a single fast proton is emitted. The criterion for the selection of such events was the approximate agreement between the angle of emission of the proton and the ionization produced by it, which follows from the requirements of the conservation of energy and momentum.

In the reactions of meson production we identified only the events of the production of charged π-mesons the ionization of which approximates the minimum.

The cross sections (in the units 10^{-27} cm^2) obtained in the various processes are as follows:

Total cross section of interaction (σ_t)	150± 15	(97)
Inelastic processes (σ_t)	99 ±12	(64)
Elastic scattering (σ_t)	51 ± 9	(33)
Absorption with change in charge $(\sigma_\alpha + \sigma_{ex})$	31 ± 7	(20)
Inelastic scattering with no change in charge(σ_i)	65 ±10	(42)
Absorption by $(p\text{–}p)$–pair $(\sigma_{\alpha(pp)})$	3 ± 2	(2)
Meson production (σ_π)	3 ± 2	(2)

The figures on the brackets indicate the number of the corresponding events of interaction. In both observed events of meson production creation of positive mesons was taking place.

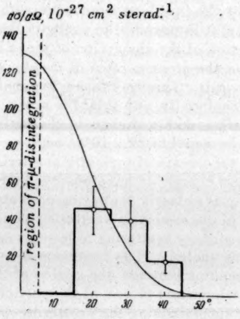
From experiments on the interaction between mesons of lower energies and complex nuclei[1,2], it is known that in the nonradiative absorption of pions by nuclei essentially only a pair of nucleons participate. Therefore, we may consider that at the energy of 330 mev the absorption of pions will all the more be accomplished essentially by pairs of nucleons. Taking into account this condition and assuming that the absorption by the $(p\text{–}p)$–pair and by the $(n\text{–}p)$–pair proceeds with equal probability, it is possible to evaluate the total cross section of the absorption of pions by helium nuclei from the cross section of the absorption by the $(p\text{–}p)$–pair. This was found to equal 9×10^{-27} cm^2. In conformity with this, the cross section of the scattering with a change in charge (recharge?) will then be equal to 22×10^{-27} cm^2.

Among the events of inelastic scattering with no change in charge we have observed three events of quasi-elastic scattering on protons and 21 events of quasi-elastic scattering of neutrons with no secondary scattering of mesons or nucleons in the same nucleus. The first were identified from the relationships between the angles of the scattering of mesons and of one of the particles of fissioning approximating the scattering on free nucleons. The second group of events was characterized by the emission of an H^3 fissioning nucleus with a comparatively short path and a momentum of the nucleons in the α-particle; moreover neither the direction of emission of the nucleus, nor the magnitude of its momentum depend on the scattering angle of the meson.

The mean differential cross sections of the inelastic scattering for the three ranges of angles in the units 10^{-27} cm^2 /sterad in a laboratory system of coordinates are as follows:

0—60°	60—120°	120—180°
7.9 ± 2.0	2.5 ± 0.8	4.5 ± 1.5

The angular distribution of the elastic scattering in a center-of-mass system is given in the Figure. Attention is drawn to the fact that not a single event of elastic scattering was observed in the angle range 5–15°. Above it was noted that the identification of elastic scattering events at small angles is associated with considerable difficulties, therefore it would be natural to assume that the latter state of affairs may be due to omissions in the analysis. In order for the differential cross section in the region 5 –15° to remain at the level 50×10^{-27} cm^2/sterad, one would expect to find here 5–6 events on the basis of the existing statistical material. A thorough second examination of 40% of all the photographs did not disclose a single event of elastic scattering (additional to the first analysis). The small statistical material does not provide the possibility of drawing a completely definite conclusion regarding the course of the angular distribution in this region, however it appears probable that in the region of small angles the differential cross sections of the elastic scattering change non-monotonically.



The general character of the angular distribution of elastic scattering can be qualitatively described within the frame of the optical model of a nucleus. The calculated angular distribution is shown by the solid curve in the Figure. The mean free path of the π-mesons and the mean potential inside the nucleus V_0, used in these calculations, were determined from the total cross sections of the elastic and inelastic scattering and for a nucleus of radius $R = (\hbar/\mu c)\, A^{1/3}$ were found to be $(2.7 \pm 0.3) \times 10^{-13}$ cm and (32 ± 8) mev respectively. The angular distribution obtained in terms of the optical model shows considerable deviation from experimental data only in the region of small angles.

If non-monotonic change in the differential cross sections of elastic scattering in the region of small angles actually takes place, then for its explanation one may draw on the interference between the coulombic and the nuclear interaction. In this case it would be necessary to consider that the amplitudes of the coulombic and nuclear scatterings of the negative pions on nuclei have different signs, in contrast to the results for low energies (less than 200 mev) where the signs of the corresponding amplitudes are the same.[2]

The calculations of the energy dependence of the mean potential inside the nucleus, carried out in the investigations[3,4] on the basis of the properties of the scattering of pions on free nucleons, are in agreement with this fact.

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Note on Waves in a Homogeneous Magnetoactive Plasma

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WITHIN recent years there appeared a number of articles by Piddington[1-3] devoted to a consideration of the properties of normal waves propagated in a homogeneous plasma situated in a magnetic field H_0. In these works the calculation of thermal motion is made by the approximation method based on equations for mean particle velocities. Such a quasi-hydrodynamic method of investigation is not new; it has been repeatedly used in the analysis of similar problems by other authors (see, for example, Refs. 4 and 5). But at the same time it should be noted that many of the problems touched upon in Refs. 1-3 have been studied on the basis of the more rigorous kinetic theory method[6-11]. Most of the studies of the latter type apparently were not known to Piddington.

Here we intend to dwell on only one of the aforementioned works by Piddington[1] in which high-frequency waves* (extraordinary, ordinary, plasma) are dealt with. We shall endeavor to elucidate in greater detail than is done in Ref. 1 the question of the relationship existing between the different types of normal waves, and then to contrast very briefly the results obtained by the quasi-hydrodynamic and the kinetic methods of investigation. From the equations of electrodynamics and from the quasi-hydrodynamic equations for electron motion with due regard to electron pressure [assuming the departures from equilibrium values to be small and proportional to $e^{i(\omega t - \mathbf{kr})}$], we can get the expression

$$\begin{aligned}\beta_e^2 (1 - u\cos^2\alpha)\, n^6 - [1 - u - v \\ + uv\cos^2\alpha + 2\beta_e^2 (1 - v - u\cos^2\alpha)]\, n^4 \\ + [2(1-v)^2 - u(2 - v - v\cos^2\alpha) \\ + \beta_e^2 (1 - 2v + v^2 - u\cos^2\alpha)]\, n^2 \\ + (1 - v)[u - (1-v)^2] = 0,\end{aligned} \tag{1}$$

where $n = ck/\omega$ is the refractive index of the waves, $u = \omega_H^2/\omega^2 = (eH_0/mc\omega)^2$ (ω_H, the electron gyrofrequency, e and m, the charge and mass of the electron), $v = 4\pi e^2 N/m\omega^2 = \omega_{0e}^2/\omega^2$ (ω_{0e}, the Langmuir frequency, N, electron concentration), α is the angle between the magnetic field H_0 and the direction of propagation k, $\beta_e = \sqrt{\varkappa T/mc^2}$ is the ratio of the mean thermal velocity of the electrons to the velocity of light c. We add that Eq. (1) was derived and discussed in a dissertation by the author of this letter as far back as 1953[12]. Reference 12 also contains an analysis of the problem by the kinetic theory method, which analysis appeared in Refs. 10 and 11 as well.

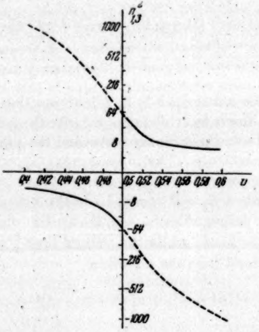
Turning to a consideration of Eq. (1) it is necessary to keep in mind that in the present nonrelativistic case, $\beta_e^2 \ll 1$. This inequality is easily satisfied in the cases that are of greatest interest from the standpoint of possible application, namely, those of the ionosphere ($\beta_e^2 \sim 10^{-7}$) and of the solar atmosphere ($\beta_e^2 \sim 10^{-4} \div 10^{-5}$). Let us examine the wave behavior in the particular cases where $\alpha = 0$ and $\alpha = \pi/2$. When propagation is in the direction of the field H_0 ($\alpha = 0$) we obtain from (1) the expression

$$n_{1,2}^2 = (1 - v)/(1 \pm \sqrt{u});\quad n_3^2 = (1 - v)/\beta_e^2, \tag{2}$$

in which the subscripts 1, 2, 3 indicate that the values for the refractive index refer to the extraordinary wave, the ordinary wave and the plasma wave, respectively. The plasma wave appears in the equation when thermal motion is taken into account (when $\beta_e \to 0$ $n_3^2 \to \infty$). In the case of transverse propagation ($\alpha = \pi/2$) we find from Eq. (1) that $n_2^2 = 1 - v$, while the values for n_1^2 and n_3^2 are determined from the equation

$$\beta_e^2 n^4 + [(v - 1)(1 + \beta_e^2) + u]\, n^2 + (v - 1)^2 - u = 0. \tag{3}$$

To ascertain the character of wave behavior it is quite usual and important to analyze the curves $n^2(v)$ by assigning fixed values to u and β_e for the given case. Considering the case where $\alpha = 0$, we see that for each type of wave there is a particular dispersion curve determined from the formulas given in (2). But the case $\alpha = 0$ is an exceptional one, and when $\alpha \neq 0$ it becomes impossible for all values of n^2 to distinguish by means of a single continuous curve the behavior of only one type of wave. In the accompanying diagram are shown curves representing $n^2(v)$ when $\alpha = \pi/2$. In contrast to the case $\alpha = 0$, the values for $n_3^2(v)$ are represented here not by a separate curve but by what appears to be a continuation of the curve $n^2(v)$. In the diagram we refer to that part of the curve $n^2(v)^1$, with $n^2 > 0$, where $v > 1 - u$, to the extraordinary wave, and the part where $v < 1 - u$, to the plasma wave. Such a separation is based on the premise that in the absence of thermal motion ($\beta_e = 0$) the plasma wave should disappear, but at the same time, as is known, the dispersion curve runs to the right of point $v = 1 - u$, and when $v = 1 - u$: $n_1^2 \to \infty$. Nevertheless, it should be emphasized that there is a certain artificiality about the above separation into two types of waves, the truth of which is made especially clear in the case involving the point $v = 1 - u$, where there is no reason for referring the n^2 values either to the plasma or to the extraordinary wave. In addition, it will be noted that similar peculiarities arise not only when $\alpha = \pi/2$, but also in the case of other values of $\alpha \neq 0$. The case where the values for α are small, and which is of some interest, is discussed in Ref. 13.



Solid curves – n_1^2 (extraordinary wave); Dotted curves – n_3^2 (plasma wave). $u = 0.5$; $\beta_e^2 = 10^{-4}$.

If, for $u < 1$, the previously discussed coupling between the plasma wave and the extraordinary wave is always present, then upon satisfaction of the condition $u \cos^2 \alpha > 1$, the plasma wave is characterized by a single continuous curve $n^2_{2,3}(v)$ along with the ordinary wave. Here, a study of the roots of Eq. (1) shows that in the vicinity of point $v = (u - 1)/(u \cos^2 \alpha - 1)$ (when $\beta_e^2 \ll 1$) there is a region in which the values for n^2 on the $n^2_{2,3}$ curve mentioned above are complex. If, for values $n^2 > 0$, the normal waves are characterized by the presence of propagation alone, and for the values $n^2 < 0$ they are undergoing pure decay, we have here an intermediate case where the propagation of the wave is associated with its attenuation in space even when collisions are disregarded in the equations. As for the case where $u < 1$, it can be established from Eq. (1) (provided $\beta_e^2 \ll 1$) that the n^2 values hold true for all the wave types.

An investigation of the problem dealt with in the preceding is entirely possible also on the basis of the kinetic theory method. The kinetic treatment leads to the establishment of the possibility of decay which is essentially due to the influence of particle thermal motion on wave propagation. This mechanism, however, is ineffective for the inequalities[9,12]

$$|(\beta_e^2 n^2 / u) \sin^2 \alpha| \ll 1, \quad |\beta_e^2 n^2 \cos^2 \alpha| \ll 1. \tag{4}$$

For slowly decaying waves we can obtain an equation which is analgous to the quasi-hydrodynamic Eq. (1)[10,11]:

$$\begin{gathered}
\beta_e^2 v\,[A \sin^2 \alpha + B \sin \alpha \cos \alpha + (1 - u)\, C \cos^2 \alpha]\, n^6 \\
- [(1 - u - v + uv \cos^2 \alpha) + O_1(\beta_e^2)]\, n^4 \\
+ [2(1 - v)^2 - u(2 - v - v \cos^2 \alpha) \\
+ O_2(\beta_e^2)]\, n^2 + (1 - v)\,[u - (1 - v)^2] = 0, \\
A = \frac{\cos^2 \alpha\,(1 + 3u)}{(1 - u)^2} + \frac{3 \sin^2 \alpha}{1 - 4u}, \\
B = \frac{4 \sin \alpha \cos \alpha}{1 - u}, \quad C = 3 \cos^2 \alpha + \frac{\sin^2 \alpha}{1 - u}
\end{gathered} \tag{5}$$

The values for the roots of this equation correspond to the correct solution of the problem, provided the inequalities given in (4) are fulfilled. The magnitudes of $O_1(\beta_e^2)$, $O_2(\beta_e^2)$ in (5) are of the order of β_e^2 and are of small consequence; the fundamental difference between Eqs. (1) and (5) is the fact that the expressions preceding n^6 in both are different. This difference in particular accounts for the fact that in the kinetic treatment, the values for n^2 can be complex also when $u < 1$, and not only in the case where $u \cos^2 \alpha > 1$.

As for the plasma waves, they undergo slow decay only in the vicinity of the point $v = (u-1)/(u \cos^2 \alpha - 1)$. The plasma waves are rapidly damped, however, when $[1 - u - v + uv \cos^2 \alpha] \gg \beta_e^2$.

The author is grateful to Prof. V. L. Ginzburg for the discussion of the contents of this communication.

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* The assumption of high-frequency is equivalent to a disregard of ionic motion.

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On the Motion of Inclusions in a Solid

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CONSIDER a foreign inclusion in a solid body of infinite extent. Let the inclusion be spherical in shape, and filled with substance (liquid or gas) in which the material of the solid under existing conditions has a marked solubility. Let a constant but infinitesimal temperature gradient ∇T be maintained in the solid; we will investigate the translational motion of the inclusion under the influence of this gradient.[1,2]

It is obvious that, in a solid, the translation of an inclusion can take place only by means of the transfer of matter into the inclusion, but a hydrodynamic mechanism similar to that involved in the rise of bubbles in a liquid is excluded (we do not consider viscous flow of the crystal). In the presence of only a thermal field, the indicated transfer is connected with the difference in saturation concentrations of the solution at the cold and hot ends of the inclusion, and takes place purely by diffusion. The presence of other fields leads, generally speaking, to the appearance of other flows, (for instance, the presence of a gravitational field of intensity g can lead to convection*). Thus each element of surface surrounding the inclusion will have a velocity

$$\mathbf{v} = (D / \rho) \nabla c, \tag{1}$$

where ρ is the density of the substance comprising the solid, ∇c is the concentration gradient of this substance in the material filling the inclusion, taken near the portion of the surface under consideration, and d is the diffusion coefficient.

The concentration c, which depends, generally speaking, on the coordinates and on the time, is determined from the equations of diffusion and thermal conduction, with suitable boundary conditions:

$$\begin{gathered} \partial c / \partial t = D\,[\Delta c + (k_T / T)\,\Delta T]; \\ \partial T_1 / \partial t - (k_T / c_p)\,(\partial \mu / \partial c)_{p,T}\,\partial c / \partial t = \chi_1 \Delta T_1; \\ \partial T_2 / \partial t = \chi_2 \Delta T_2, \end{gathered} \tag{2}$$

where k_T is the coefficient of thermal diffusion, μ the chemical potential of the contents of the inclusion, c_p their specific heat, and χ_1, T_1 and χ_2, T_2 respectively the thermal conductivity and temperature inside and outside the inclusion. Leaving the boundary conditions out of the picture for the moment, we go over in these equations to a coordinate system in which the inclusion is at rest. Terms proportional to $\mathbf{v}\nabla T$ and $\mathbf{v}\nabla c$, appearing as a result of this transformation, will be of second order in ∇T (since $v \sim \nabla T$). For $\nabla T = \text{const}$, v does not depend explicitly on the time, and the partial derivatives of the temperature and concentration with respect to time will be at most of second order. Consequently, to the approximation being considered here, both the temperature and the concentration satisfy Laplace's equation. We now return to the conditions at the surface of separation. These will have the form:

$$T_1 = T_2, \quad \varkappa_1\,\partial T_1/\partial n - \varkappa_2\,\partial T_2/\partial n = -qD\,\partial c/\partial n, \quad c = c_0, \tag{3}$$

$\varkappa_1$ and $\varkappa_2$ are the respective coefficients of thermal conductivity, and $\partial/\partial n$ is an operator denoting the derivative along the normal to the surface. The right-hand side of the second condition makes allowance for the evolution (or absorption) of latent heat of crystallization q at the boundary, and the third conditions requires that the solid solution be saturated at this surface. Since the gradient ∇T is small, the change in temperature along the surface of separation will not be great, but if one reckons the temperature and concentration with respect to their values at the center of the inclusion, the last condition in (3) will be written thus:

$$c = (dc_0 / dT)\, T_1. \tag{4}$$

Here the derivative dc_0/dT is taken along the appropriate equilibrium curve. Thus at the limits of the region, the concentration is proportional to the temperature, and since both the temperature and the concentration satisfy the same equation, there will be a similar relation between them everywhere inside the inclusion (irrespective of the form of the bounding surface). Therefore at the surface of separation

$$-qD\, \partial c / \partial n = -qD\, (dc_0 / dT)\, (\partial T_1 / \partial n).$$

If we now replace $\varkappa_1$ by $\varkappa_1 + q\,dc/dT$ in condition (3), we obtain the well-known problem concerning the distribution of temperature around a stationary spherical object for a constant temperature gradient at infinity (see, for example, Ref. 3). Multiplying its solution over the interior of the region by dc_0/dT, we obtain the distribution of concentration over the inclusion. Making use of Eq. (1), we obtain finally

$$\mathbf{v} = \frac{3\varkappa_2 D}{\varkappa_1 + 2\varkappa_2 + qD\, dc_0/dT}\, \frac{dc_0}{dT}\, \frac{\nabla T}{\rho}. \tag{5}$$

From this equation it is obvious that the velocity of translation of the inclusion does not depend on its dimensions.

The value of **v** is determined by the quantity dc_0/dT, which under conditions of constant pressure is equal to qc_0/kT^2. In substances whose solubility in the material filling the inclusion does not increase with temperature, the inclusion moves in the direction of the temperature gradient. If however $dc_0/dT < 0$, the inclusion has to translate in the opposite direction. It is clear that the process under investigation is governed by Le Chatelier's principle. Indeed, for $q > 0$, the diffusion at the hot end of the inclusion and the condensation at the cold end both represent the same process, striving to bring the system back into equilibrium; this equilibrium is constantly being destroyed, however, by the source maintaining the temperature gradient.

It is not difficult to see that an analogous process takes place for the transfer of matter in a liquid medium from a crystal having a high temperture to a crystal with a lower one; that is, for the growth of one crystal at the expense of another.

The coefficient of diffusion has the order of magnitude 10^{-5} cm^2/sec; $dc_0/dT \sim 5\times 10^{-2}$ g/cm^3–sec–deg; $q \sim 10^2$ cal/g; $\varkappa_2 \sim 5 \times 10^{-3}$ cal/cm–sec–deg.

Consequently, $qD\, dc_0/dT \ll \varkappa_2$, and

$$\mathbf{v} = (3D / 2\rho)\, (dc_0 / dT)\, \nabla T,$$

that is, **v** does not depend on the thermal conductivity of the solid or of the contents of the inclusion.

If the conditions under which an inclusion exists are critical conditions for its contents, then the coefficient of diffusion will fall to zero and the inclusion will come to rest. For a temperature gradient of the order of 1deg/cm, the velocity of translation of the inclusion amounts to about 0.1 mm/day. If conditions are created whereby matter can be transferred convectively, the process ought to be noticeably accelerated.

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*It is clear, for example, that there will be convection if the vectors **g** and ∇T are parallel, but not if they are antiparallel.

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Investigation of High Energy Electron Showers by the "Emulsion Chamber" Technique

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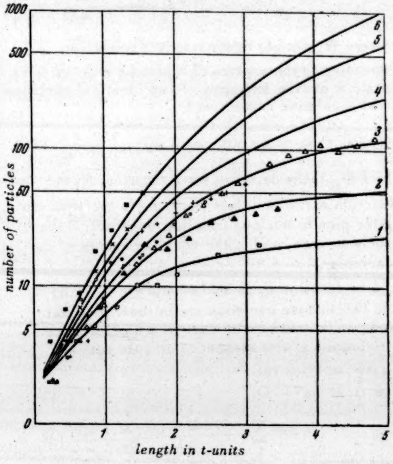
IN the emulsion chamber suggested in Ref. 1, we have studied the development of electron showers of high energy ($E = 10^{12}$ ev) at small depths (2–3 τ–units). The results obtained are compared with cascade theory.[2,3]

The emulsion chamber consisted of 24 emulsion plates (NIKFI) type R with emulsion thickness of 100 μ put together into a stack and held in a special frame. Iron plates of thickness 3.5 mm were placed between emulsions. The distance between adjacent layers of emulsion was 5 mm. The glass on which the emulsion was poured was previously polished to a size of 86×116 mm^2 with an accuracy of 0.05 mm. This allowed the exact placement of adjacent plates which is essential in following a shower.

The emulsion chamber was exposed in the stratosphere ($\sim$ 4 hrs. at height $\geq$ 20 km). After development the plates were divided into two groups for convenience during scanning. Two plates from the middle of the chamber were scanned with magnification of $20 \times 7 \times 1.5$. During this time showers were noted which had more than 5–7 parallel tracks in a radius of 100 μ which thereafter continued into the upper and lower plate. With the available accuracy of finding the continuation of a shower (approximately 200 μ) and a track background of 160 mm^{-2} it was impossible to follow a single relativistic particle or shower, consisting of several tracks, through considerable gaps ($\geq$ 400 μ).



Comparison of the experimental data on the development of electron showers with cascade curves. *1*-$y = 5$ ($E_0 = 1.8 \times 10^{11}$ ev); *2*–$y = 6$ ($E_0 = 7 \times 10^{11}$ ev); *3*– $y=7$ ($E_0 = 10^{12}$ev); *4*–$y=8$ ($E_0 = 6 \times 10^{12}$ ev); *5*–$y= 9$ ($E_0 = 9 \times 10^{12}$ ev); *6*–$y = 10$ ($E_0 = 27 \times 10^{13}$ ev)

A shower arising in the emulsion chamber from a single photon is characterised by a gradually decreasing number of particles until two tracks are seen at very small (2–3 μ) distances from each other, or a single track with double ionization appears. It is impossible to follow the shower further as indicated above. It is essential to note that in following a shower the only tracks considered as a part of the shower have direction parallel to the axis of the shower. The chief contribution to deflection of tracks from the parallel is multiple scattering. The average angle of multiple scattering for particles of energy e passing through a distance t is given by the relation

$$\langle \vartheta \rangle = (E_s / E) \sqrt{t},$$

where $E_s = 21$ mev and t is expressed in radiation units. In the present paper the parallelism of tracks was determined to within one degree. This means that in each shower we registered particles with energy $\geq 10^9$ ev.

During the microscopic examination of the emulsion chamber 8 electron proton showers were found. For each of these the total number of particles in all emulsion plates was calculated and the dependence of the number of particles on the depth was found. This data was compared with the cascade curves obtained from the calculations of Arli[2] for depths up to $2t$–units and from the calculations of Belenkov[3] for depths greater than $2t$–units for various values of the energy of the initial electron with secondary particle energy greater than $E_1 = 10^9$ ev. Since the observed showers began with the appearance of the first electron pair the shower is considered to be caused by two primary electrons of equal energy. Curves calculated taking this into account are shown in the diagram. The experimental points corresponding to various showers are shown by the different symbols.

At distances > 1 t–unit from the beginning of the shower the dependence of the number of particles on the depth coincides with one of the cascade curves calculated for a particular energy of the initial electron. At depths less than one t–unit the number of particles in the shower fluctuates strongly.

The data given in the Figure shows that there is fairly good agreement between the experimental results and cascade theory for electron energies $E_0 \sim 10^{12}$ ev for small thicknesses of matter. Also in the observation of the electron showers at depths $\geq 2t$–units it is possible to evaluate the initial electron energy from the cascade curves to within a factor of 2 or 3. The results of these evaluations are shown in the table.

In conclusion the author would like to express deep thanks to I. L. Rosental and D. S. Chernavskii for their consideration of the experimental results and to N. A. Dobrotin for valuable comments and to R. M. Grizunova for the scanning.

TABLE

No. of Shower	1	2	3	4	5	6	7	8
Total length in t-units . .	2.4	3.4	1.7	3.2	3	4.6	1.6	1.5
$y = \ln(E_0/E_1)$	7	6	7	5	7	7	9	10

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Translated by G. L. Gerstein
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Dispersion Relations for Pions Scattered by Deuterons

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CONSIDER zero-angle elastic scattering of pions by deuterons. The dispersion relations corresponding to this process differ substantially from the dispersion relations for the scattering of pions by free nucleons: first, the dispersion relations depend on the polarization of the deuterons; second, if the Coulomb interaction is neglected, there is only one dispersion relation for the sum of the scattering amplitudes of the positive and negative pions.

Let us denote by $D_m(\omega)$ and $A_m(\omega)$ the real and imaginary parts of the amplitude of the scattering of pions with energy ω by deuterons having a spin projection m along the direction of motion of the pions. Using the relationship between the imaginary part of the zero-angle scattering amplitude and the total cross section, $A_m(\omega) = (k/4\pi)\sigma_m(\omega)$ (where $k^2 = \omega^2 - \mu^2$, and μ is the pion mass), and using the usual procedure for obtaining the dispersion relations[1,2] we obtain

$$D_m(\omega) - D_m(\mu) = \frac{2k^2}{\pi}\int_0^{\mu} \frac{\omega' A_m(\omega')\,d\omega'}{k'^2(\omega'^2 - \omega^2)} + \frac{k^2}{2\pi^2}\int_0^{\infty} \frac{\omega'\sigma_m(\omega')\,d\omega'}{(\omega'^2 - \omega^2)\,k'}. \quad (1)$$

To determine the contribution from the region $0 \le \omega \le \mu$ we employ the following expressions for $A_m(\omega')$, obtained readily from Ref. 1:

$$A_m(\omega') = \pi \sum_f |M_m(\omega', \mathbf{f})|^2 \,\delta\left[\omega' - \varepsilon_0 - \frac{k'^2}{4M} - \frac{f^2}{M}\right], \quad (2)$$

where $M_m(\omega', \mathbf{f})$ is the matrix element corresponding to the capture of a pion by a deuteron in state m and the formation of two identical nucleons with a relative momentum $\mathbf{f}$:

$$M_m(\omega', \mathbf{f}) = (\sqrt{2}\,g/M)\,\langle \Phi_m^* \sigma_1 \mathbf{k}' F_i(\mathbf{k}', \rho)\,\Psi_{\mathbf{f}} \rangle. \quad (2')$$

here Φ_m is the deuteron wave function, $\Psi_{\mathbf{f}}$ the wavefunction of two identical nucleons in the final state, g the pion to nucleon coupling constant, M the nucleon mass, and ϵ_0 the coupling energy of the deuteron.

The function $F_i(\mathbf{k}', \rho)$ equals $\sin(\mathbf{k}'\rho/2)$ if the two forming nucleons are in the triplet state and $\cos(\mathbf{k}'\rho/2)$ in the case of a singlet state.

Integrating with respect to ω', we get

$$\frac{2k^2}{\pi}\int_0^{\mu} \frac{\omega' A_m(\omega')\,d\omega'}{k'^2(\omega'^2 - \omega^2)} = 2k^2 \sum_{\mathbf{f}} \frac{\mu^2/4M - (f^2/M + \varepsilon_0)}{\tilde{k}^2(\omega^2 - \tilde{\omega}^2)}\,|M_m|^2. \quad (3)$$

Generally speaking, $\tilde{\omega}$ is a function of f, determined by the conservation laws. However, taking into account the fact that the matrix element differs substantially from zero only in the region $\mathbf{f} \sim \mathbf{k}/2$, we shall assume hereinafter $\tilde{\omega} \approx \mu^2/2M - \epsilon_0$. Next, since small changes in ω' correspond to large changes in f, we extend the summation with respect to f to infinity. Using the completeness of the set of functions $\Psi_{\mathbf{f}}$, we have

$$\sum_f |M_m|^2 = \frac{2g^2}{M^2}\,\tilde{k}^2 \left\{\delta_{m1}\int \varphi_0^2 F_t^2\,d\rho + \delta_{m0}\int \varphi_0^2 F_s^2\,d\rho\right\}, \quad (4)$$

where $\varphi_0\ (\rho)$ is the coordinate portion of the deuteron wave function.

The second term in expression (3) is calculated in the following manner:

$$\sum_{\mathbf{f}} (f^2/M+\varepsilon_0)\,|M_m|^2 \qquad (5)$$

$$=\frac{2g^2}{M^2}\sum_{\mathbf{f}}\langle\Phi^*_m\sigma_1\mathbf{k}F_i\Psi_{\mathbf{f}}\rangle\langle\Psi^*_{\mathbf{f}}(H_1\sigma\mathbf{k}F_i-\sigma_1\mathbf{k}F_iH)\,\Phi_m\rangle,$$

where H is the Hamiltonian of the interaction of the two nucleons. Its general form (without taking the tensor forces into account) is as follows:

$$H=\mathbf{f}^2/M+{}^1/_4\,U_t(\rho)\,(\sigma_1\sigma_2+3) \qquad (6)$$

$$-{}^1/_4\,U_s(\rho)\,(\sigma_1\sigma_2-1)$$

$$+\{{}^1/_4\tilde{U}_t(\rho)\,(\sigma_1\sigma_2+3)-{}^1/_4\tilde{U}_s(\rho)\,(\sigma_1\sigma_2-1)\}\,P_{12}.$$

Here $\mathbf{f}^2/\mathrm{M}$ is the kinetic-energy operator, U_t and U_s are the potential energies in the triplet and singlet states, $\tilde{\mathrm{U}}_t$ and $\tilde{\mathrm{U}}_s$ the exchange energies, and P_{12} is the particle commutation operator.

Calculating the sum (5) with the aid of the Hamiltonian (6), and inserting the result into (1), leads to the following dispersion relationships.*

For deuterons polarized parallel (anti-parallel to the incident beam:

$$D_{\pm 1}(\omega)-D_{\pm 1}(\mu) \qquad (7')$$

$$=g^2\left(\frac{\mu}{2M}\right)^2\frac{2}{M}\,\frac{k^2}{\omega^2-\tilde{\omega}^2}\left[1+\frac{8M}{\mu^2}\int\varphi_0^2\sin^2\frac{\tilde{\mathbf{k}}\rho}{2}\,\tilde{U}_t\,d\rho\right]$$

$$+\frac{k^2}{2\pi^2}\int\limits_{\mu}^{\infty}\frac{\omega'\sigma_{\pm 1}(\omega')\,d\omega'}{k'(\omega'^2-\omega^2)}$$

For deuterons polarized perpendicular to the incident beam:

$$D_0(\omega)-D_0(\mu) \qquad (7'')$$

$$=g^2\left(\frac{\mu}{2M}\right)^2\frac{2}{M}\,\frac{k^2}{\omega^2-\tilde{\omega}^2}\left[1+\frac{4M}{\mu^2}\int\varphi_0^2\cos^2\frac{\tilde{\mathbf{k}}\rho}{2}\,\{\tilde{U}_s-U_s\right.$$

$$\left.+\tilde{U}_t+U_t\}\,d\rho\right]$$

$$+\frac{k^2}{2\pi^2}\int\limits_{\mu}^{\infty}\frac{\omega'\sigma_0(\omega')\,d\omega'}{k'(\omega'^2-\omega^2)}.$$

The dispersion relations obtained for the scattering of pions by deuterons contain, in addition to the constant g, also certain effective values of the potential interaction energy of two nucleons in different states and these values affect substantially the value of the singularity term for deuterons, polarized perpendicularly to the incident beam.

The authors express their gratitude to Academician L. D. Landau for valuable comments.

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*Let us call attention to the fact that we obtained (7′) and (7″) without using the actual form of the coordinate part of the wave function of the deuteron.

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Influence of the Earth's Magnetic Field on the Space-Distribution of Particles in Extensive Air Showers

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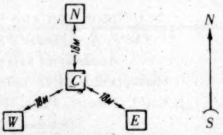
THE space-distribution of charged particles in the extensive air showers of cosmic rays has been studied in a number of experiments[1-4]. In none of the computations used, despite their high statistical accuracy, has the influence of the earth's magnetic field been taken into account. The theoretical estimates made[5] have shown that the distortion of axial symmetry in the space-distribution of the electron component of the extensive air shower, produced by the action of the earth's magnetic field, does not exceed the statistical limits of experimental errors. Nevertheless, the results of the experimental investigation given in Ref. 6 aroused doubts as to whether the disregard of the influence of the earth's magnetic field on the space-distribution of shower particles is justified. This disregard, by the way, was considered permissible in Ref. 2.

Subsequently, we made a supplementary analysis of the experimental data given in Ref. 2. These related to a study completed in the summer of 1952 at an elevation of 3860 m (Pamir). The relative position of a part of the experimental set-up with respect to the earth's magnetic field is shown in diagram *a*. The letters *C*, *N*, *E*, *W* (central, north, east, west) indicate groups of counters connected to a hodoscope (counter-telescope). The cases under consideration were those in which the axis of the extensive air shower passed at a distance of not more than 5 m from the center of the set-up *C*. The influence of the earth's magnetic field on the space-distribution of shower particles can be described by the relationship

$$\Delta = 2(p_L - p_M)/(p_L + p_M).$$

Here p_M is the mean flux density of the charged particles above the hodoscopic group of counters *N*, p_L is the mean flux density of the particles above the groups *W*, *E*. For the average distance ~ 18 m from the shower axis the value Δ_{18} equals 0.00 ± 0.03 and, therefore, the disregard of the influence of the earth's magnetic field on the space-distribution of shower particles in study[2] is fully justified.

In the summer of 1955, working at an elevation of 3860 m (Pamir), we continued our investigation of the influence of the earth's magnetic field on the space-distribution of shower particles. The layout of the set-up is shown in diagram *b*. At the center of the set-up, *C* was a controlling counter group consisting of three counters placed under a shield of lead 6 cm thick, and another counter 3 m away from the shielded counters. The counters were included in the circuit of a fourfold coincidence arrangement. As shown in Ref. 1, such a control system serves chiefly as a means of registering the cores of the extensive air showers. Counters of different areas, connected to a large hodoscopic device, made it possible to determine the flux density of the particles at the center of the set-up for each registered shower. Similar hodoscopic groups were used at a distance of 100 m to the north, south, east and west of the center of the set-up (*N*, *S*, *E* and *W*, respectively).



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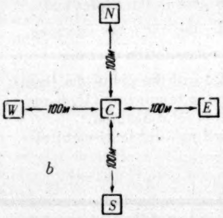

b

The primary energy of the showers studied in the most recent investigation differed little from the primary energy of the showers studied in 1952[2], and amounted to ~ $10^{14} - 10^{15}$ ev. A study of more than 800 extensive air showers whose axes were not more than 10 m away from the center of the set-up enabled us to establish the value of Δ for distances of 100 m from the shower axis, namely, $\Delta_{100} = 0.12 \pm 0.03$. In the case under discussion, the symbol p_M was used to denote the mean flux density of the particles at points *N* and *S*.

Our experimental data pertaining to the influence of the earth's magnetic field on the space-distribution of charged particles in the extensive showers agree with the findings given in Ref. 7 by Norman, who failed to trace the forementioned influence in the central part of the shower. At the same time, however, it is difficult to reconcile our findings with the results obtained by Chaloupka and Petrzilka in their investigation[6], where the value of Δ for distances ~ 30 m is greater than or equal to 0.35 ± 0.15.

The asymmetrical effect, revealed in our experiments, in the distribution of the flux density of the charged particles at a distance ~ 100 m from the axis of the extensive air shower is equivalent to a 6-10% enlargement of the shower radius in the W-E direction at the stated distance from the axis, which is approximately twice as high as the figure arrived at in the theoretical estimates in Ref. 5. In our opinion, the use of the theoretical values offered by Cocconi[5] results in a lower figure because in his computations the latter fails to take into account the regular deflection of low energy electrons (< 100 mev) in the air layer adjacent to the observation level. An approximate evaluation of the deflection of these electrons, which are prevalent at great distances from the shower axis, eliminates the discrepancy between the theoretical estimates given in Ref. 5 and the results of our investigations.

In conclusion, the authors express their gratitude to V. M. Seleznev, V. V. Krugovykh, I. F. Maklakova and other co-workers.

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On a Certain Regularity of Decaying Unstable Particles

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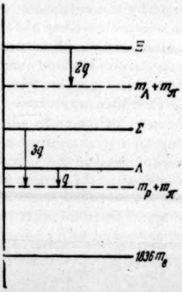
At the present time the following values of the masses of stable and unstable particles have been firmly established as universal (in electron masses): $m_\nu = m_\gamma = 0$; $m_e = 1$; $m_\mu = 207$; $m_\pi = 274$; $m_K = 966 \pm 3$; $m_p = 1836$; $m_\Lambda = 2181 \pm 1$; $m_\Sigma = 2327 \pm 3$; $m_\Xi = 2585 \pm 15$.

From among these particles, μ, π, K, Λ, Σ and Ξ are unstable. The values of the decay energy Q, experimentally observed or computed from the known masses of the particles and from decay schemes, are given below (in mev):

$\pi^0 \to 2\gamma$	$Q = 135$	$n = 3.8$
$\mu \to e + 2\nu$	$Q = 106$	$n = 3.0$
$\pi \to \mu + \nu$	$Q = 34.5$	$n = 1.0$
$K \to 3\pi$	$Q = 75.0 \pm 1.5$	$n = 2.1$
$K \to 2\pi$	$Q = 214 \pm 5$	$n = 6.0$
$K \to \mu + \nu$	$Q \sim 389$	$n = 11.0$
$K \to \mu + \pi^0 + \nu$	$Q \sim 248$	$n = 7.0$
$\Lambda^0 \to p + \pi^-$	$Q = 37.0 \pm 1.0$	$n = 1.0$
$\Sigma \to n + \pi$	$Q = 111.0 \pm 3$	$n = 3.1$
$\Xi \to \Lambda^0 + \pi$	$Q = 66 \pm 6$	$n = 1.9$

The third column contains the quantity $m = Q/q$, where $q = 35.5$ mev $= 69.5\, m_e$. All values of n are quite close to integers. An exception is noted only in several cases, when the decay leads only to stable particles (for example, neutron or π^0).

The kinetic energy liberated in the decay of unstable particles is thus a multiple of 35.5 mev. The experimentally observed energy-level scheme for hyperons* is shown in the diagram.



If the above statements are correct and if new unstable particles exist, they should be located among the mass numbers M satisfying the relationship

$$M - (m_p + nm_\pi) = n_1 q$$

for particles heavier than protons, or the relationship

$$M - nm_\pi = n_1 q$$

for mesons heavier than the pion. In these equations n and n_1 are integers.

It is interesting to note that the number of electron masses entering into q is very close to the value $1/2\ \alpha = 68.5$.

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* Incidentally, several possible hyperon models exist, but there is no point in dwelling on them in this note.

Translated by J. G. Adashko
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Peculiarities of the Temperature Dependence of the Electrical Resistance of Ferromagnetic Metals at Low Temperatures

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FROM general considerations it follows that for ferromagnetic metals we can expect the appearance of a peculiarity in the temperature dependence of the electrical resistance at low temperatures, brought about by the collisions of the conduction electrons withthe carriers of ferromagnetism. In order to make clear this peculiarity, measurements were carried out on polycrystalline samples of iron, nickel and platinum in the temperature interval from 4.2° to 1.23° K. Platinum was chosen for comparison of the ferromagnetic metals with a metal of the transition group that was nonferromagnetic.

The iron and nickel used in the investigation were from Hilger and were the purest at our disposal. From these we prepared specimens in the form of thin ribbons. The residual resistance of the iron specimen amounted to $R_{4.2^\circ\mathrm{K}}/R_{0^\circ\mathrm{C}} = 3.9328 \times 10^{-2}$, where $R_{0^\circ\mathrm{C}} = 0.5091$ ohm, and of the nickel, $R_{4.2^\circ\mathrm{K}} / R_{0^\circ\mathrm{C}} = 1.0148 \times 10^{-2}$, where $R_{0^\circ\mathrm{C}} = 0.8407$ ohm. The specimen of platinum was a resistance thermometer with residual resistance $R_{4.2^\circ\mathrm{K}} / R_{0^\circ\mathrm{C}} = 3.6805 \times 10^{-3}$ and $R_{0^\circ\mathrm{C}} = 59.79487$ ohms.

The measurements on platinum showed that the curve of the temperature dependence of the resistance of the platinum was accurately described by the expression

$$R_T / R_{0^\circ\mathrm{C}} = (R_{0^\circ\mathrm{K}} / R_{0^\circ\mathrm{C}}) + BT^2,$$

where

$$B \approx 1.8\cdot 10^{-6}, \quad R_{0^\circ\mathrm{K}} / R_{0^\circ\mathrm{C}} = 3.6486\cdot 10^{-3}.$$

The results of measurement on the iron and nickel are shown on the graphs of Fig. 1 and Fig. 2, where the magnitude of the relative resistance $R_T / R_{0^\circ\mathrm{C}}$ is plotted along the ordinate and the temperature in degrees Kelvin is plotted along the abscissa. The different symbols for the points correspond to different series of measurements on one and the same sample.

For iron, the resistance curve (Fig. 1) cannot be described simply by a quadratic function, as is the case for platinum. Its behavior is accurately expressed by a binomial in T^2 and an additional linear term:

$$R_T / R_{0^\circ\mathrm{C}} = (R_{0^\circ\mathrm{K}} / R_{0^\circ\mathrm{C}}) + AT + BT^2.$$

where $A = (4 \text{ to } 4.9) \times 10^{-6}$, $B = (1 \text{ to } 1.2) \times 10^{-6}$; $R_{0^\circ\mathrm{K}} / R_{0^\circ\mathrm{C}} = 3.9293 \times 10^{-2}$.

The temperature dependence of the resistance for nickel is plotted in Fig. 2 on the same coordinates which were used to describe iron. We obtain the formula

$$R_T/R_{0^\circ\mathrm{C}} = (R_{0^\circ\mathrm{K}}/R_{0^\circ\mathrm{C}}) + AT + BT^2,$$

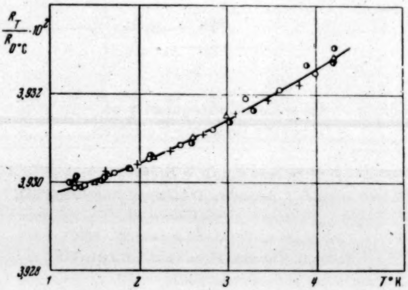

FIG. 1

in which $A = (0.8 \text{ to } 2.2) \times 10^{-6}$; $B \approx 2.7 \times 10^{-6}$; $R_{0^\circ\mathrm{K}} / R_{0^\circ\mathrm{C}} = 1.0086 \times 10^{-2}$.

For nickel, the linear component is smaller in magnitude and is determined less acuurately. Here it is necessary to separate a small value against a background of the much stronger quadratic dependence of the resistance. More accurate data on the values of the linear components of the electrical resistance in iron and nickel can be obtained with measurements on single crystals and at temperatures below 1° K.

In this fashion a peculiarity is observed in the temperature dependence of the electrical resistance

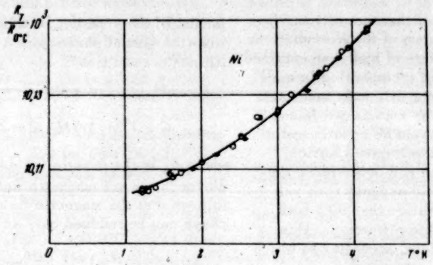

FIG. 2

of iron and nickel, which distinguishes them from other metals and which manifests itself in the presence of a linear term in the temperature dependence of the resistance.

It is interesting to note that the reults of the present research agree with the conclusions obtained in one of the works of Turov.[1]

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Translated by R. T. Beyer
104

The Crystalline Structure of Hydrogen and Deuterium

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THE structure of solid hydrogen has been investigated by Keesom and his co-workers[1], who found it to possess a hexagonal close-packed lattice with parameter $a = 3.75$ A. The structure of deuterium was not studied, and it was to determnne this that the present work was undertaken. Specimens of solid deuterium were obtained by condensation of the gas onto a copper capillary filled with liquid helium. Use of the strong-focus method of x-ray crystallography made it possible to obtain x-ray patterns with sharp lines for deuterium with exposures of one to two hours. Unfortunately, as a result of the rapid decrease of the atomic form factor with angle, the deuterium lines were visible only at small angles; this made it difficult to obtain reliable measurements from the x-ray patterns or to determine accurately the parameters of the lattice. With as much confidence as these x-ray patterns seemed to warrant, we determined the structure of deuterium to be tetragonal, with a ratio of axes $c/a = 0.94$ and a paramter $a = 5.4$ A. This leads to a density of 0.18 gm/cm^2 for deuterium, which differs by only 10% from the value obtained by direct measurement (2). In view of these results, it appeared advisable to review the data on the structure of hydrogen, for it seemed surprising that the two isotopes should crystallize into lattices having different symmetry. In particular, such a difference might arise from the occurrence of polymorphism in the two isotopes, with transition points in the vicinity of 4.2° K, so that at this temperature they might be found in different phases. However, x-ray patterns for deuterium and hydrogen obtained at lower temperatures failed to confirm this supposition—neither hydrogen nor deuterium alters its structure in the temperature range from 1.5° to 4.1°K.

In the paper by Keesom, *et al.*,[1] the x-ray patterns themselves are not shown; it appears, however, that they consisted of discrete reflections, through which Debye curves were drawn. A direct computation of the line width to be expected from the conditions prevailing in the experiment shows this width to exceed the separation of certain of the more closely-spaced lines; i.e., the reflections which these authors have assigned to different lines could actually belong to a single line. This is the probable explanation for the fact that the five intense lines in the x-ray patterns obtained by Keesom, *et al.*, correspond to three lines in our patterns. Moreover, certain lines are erroneously ascribed by Keesom, *et al.*, to the β-spectrum. An exposure made through a filter passing only the

β-radiation showed that all of the intense lines belong to the interference system of the $K\alpha$-radiation. Thus, the data on the structure of hydrogen obtained at the Leiden laboratory, and incorporated into all of the reference literature, are evidently incorrect. With the aid of the Hull-Davey curves, we found that the hydrogen patterns could be equally well interpreted as arising from a tetragonal lattice. That the lattices of hydrogen and deuterium correspond to crystals of non-cubic syngony receives confirmation from our observation that they both possess the property of double refraction. This does not support the older data, according to which solid hydrogen is optically isotropic.[3]

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An Experimental Manifestation of Instability Of the Normal Phase in Superconductors

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IN the experiments of Faber,[1] considerable supercooling of the normal phase of Al has been achieved near the critical temperature T_c. The magnetic field H_s at which the transition into the superconducting state takes place is approximately the same for various samples and has a mean value $H_s \sim 0.05\, H_{cm}$, where H_{cm} is the equilibrium critical field for the bulk metal. Values of $H_s \sim (0.035 - 0.04) H_{cm}$ were also observed (cf. Fig. 3 of Ref. 1). This gives rise to the impression that the limiting supercooling is a characteristic of the ideal metal, which cannot be supercooled to values of the field below some value H_{c1}, and that for Al $H_{c1} \sim (0.035 - 0.04)\, H_{cm}$.

We would like to call attention to the fact that this result follows directly from the theory of superconductivity developed in Ref. 2. Actually, it is shown in Ref. 2 that under certain conditions the normal phase of a superconductor becomes unstable with regard to the formation of lamina (nuclei) of the superconducting phase. In particular, these lamina of the superconducting phase are formed when the normal phase is in a magnetic field fulfilling the condition*

$$H = \varkappa H_{cm} / \sqrt{2}\,(n + 1/2), \qquad n = 0,\ 1,\ 2,\ 3, \ldots, \quad (1)$$

$$\varkappa = (\sqrt{2}e / \hbar c)\, H_{cm} \delta_0^2 = 2{,}16 \cdot 10^7 H_{cm} \delta_0^2,$$

where δ_0 is the penetration depth for the superconductor in a weak magnetic field. From Eq. (1) it follows that the magnetic field within the normal phase can be reduced only as far as the value

$$H_{c1} = \sqrt{2} \varkappa H_{cm}, \quad (2)$$

which is obtained from (1) for $n = 0$. In fields $H > H_{c1}$ —the formation of nuclei of the superconducting phase is associated with the appearance of a surface energy; the normal phase is therefore metastable over the range $H_{c1} < H < H_{cm}$ If, however, $H = H_{c1}$, the normal phase is unstable, and the superconducting transition must take place.** For Al near T_c, $\varkappa = 0.025$[(cf. Ref. 3, in which are given the values*** $\varkappa_0 = 2\varkappa$, $(T_c) = 0.050$)]. Hence, in accordance with (2), $H_{c1} = 0.0354\, H_{cm}$, which is in excellent agreement with the experimental value cited above. We note that for Al the theory is also in complete accordance[3] with experiment[1] with regard to the magnitude of the surface energy, as determined by this same parameter $\varkappa$. For Sn the limiting value H_{c1} is not reached. This circumstance may be connected with the fact that the case of an anisotropic metal is in general more complex. It is more probable, however, that in this case the reason is the same as that applying to Al for $T < 0.9\, T_c$, where superconductivity arises for field $H_s > H_{c1}$. In the region $T > 0.9 T_c$, however, as is shown in Ref. 1, the formation of nuclei is impeded by the fact that the characteristic length Δ exceeds the distance between the lattice "defects", which serve as nucleation centers. For Sn near T_c the length Δ is on the order of four times smaller than for Al, as a consequence of which the formation of nuclei is easier.

For metals having small values of $\varkappa$ the instability of the normal phase can be manifested only through supercooling. On the other hand, as is noted in Refs. 2 and 7 and is clear from (2), for $\varkappa > \varkappa_c = 1/\sqrt{2}$ instability of the normal phase occurs even for $\overline{H} = H_{c1} \geq H_{cm}$; superconductors for which

$\varkappa > 1/\sqrt{2}$ should therefore behave anomalously. It appears to us that this is just the effect observed in the case of the alloys of Sn with In investigated in Refs. 6, 8, and 9. For concentrations of In below about 2.5% these alloys behave almost as ideal suprrconductors; beyond this point their properties change sharply.[6,9] At the same time, the penetration depth for an alloy containing 2.5% In is roughly twice that for pure Sn (cf. Ref. 6). From this it follows that the parameter $\varkappa = 2.16 \times 10^7 H_{cm} \delta_0^2$ increases by about four times and near T_c is approximately 0.35. This is about one-half the critical value $\varkappa_c = 0.0707$. The discrepancy, however, does not appear too serious when we consider the known inaccuracy of the available data, the complexities associated with the anisotropy of tin****, and, finally, the presence of lattice defects even in identical alloys.[6,9] In view of this latter circumstance $\varkappa$ may exceed $\varkappa_c$ in the vicinity of individual defects even in specimens for which the mean value of $\varkappa > \varkappa_c$; furthermore, the condition for instability of the normal phase may be altered near the defects. From this point of view it is significant that for $\varkappa = 0.35$ the length $\Delta \approx 2\delta_0$ while for pure tin $\Delta \approx 15\delta_0$. It is therefore quite natural to suggest that the value of Δ is reduced to zero without particular difficulty near the defects. Thus, the results given in Refs. 6, 8 and 9 are in qualitative agreement with the theory of Ref. 2, although there are as yet insufficient data for a quantitative comparison (for this latter purpose it would be desirable to investigate alloys based on aluminum or some other cubic metal). The author, therefore, cannot agree with the opinion of Pippard,[6], who feels that the theory of Ref. 2 meets with difficulties in the case of superconductors. As regards the decrease in the concentration of the "superconducting electrons" $n_s = mc^2 / 4\pi e^2 \delta_0^2$ with increasing impurity concentration N_i, the explanation of this fact lies generally outside the scope of the phenomenological theory. If one relies upon the current qualitative ideas concerning the nature of superconductivity, the decrease of n_s with increasing N_i seems quite natural (the value of n_s is determined by the degree of "stiffness" of the wave function for the electronic ground state;[11] with increasing N_i the electronic structure "loosens" and n_s falls). It seems to us that Pippard's ideas[6,8] concerning the change of the "region of coherence" go no further than other such qualitative concepts; with regard to the nonlocal character of the relation between current and field they meet with objections.[12]

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*In Ref. 2, on p. 1072, this formula is given in different units (in Ref. 2, the field $H_0 = H/\sqrt{2}H_{cm}$); furthermore, in Ref. 2 Eq. (1) is considered only in the approximation to the case in which $H \gg H_{cm}$ (i.e., $H_0 \gg 1/\sqrt{2}$). This case will also be treated below.

**We note that, as has been shown in Ref. 2, the superconducting phase is metastable and, in consequence, can be superheated within the region of fields $H_{cm} < H_{c2}$. For $H > H_{c2}$, the superconductivity must be destroyed. As $\varkappa \to 0$ the fields $H_{c2} \to \infty$, and decreases with increasing $\varkappa$. Values of H_{c2} for a given $\varkappa$ may be obtained only through numerical calculation. With regard to the fields H_{c1} and H_{c2} in films (cf. Ref. 4).

***We note that the dependence of the free energy upon the concentration $n_s = |\Psi|^2$ of the superconducting atoms assumed in Ref. 3 (cf. also Ref. 5) has been confirmed for all T by the experiments of Whitehead, *et al.*, as D. Shoenberg has kindly informed us (cf. also Ref. 6).

****For a generalization of the theory of Ref. 2 to the case of anisotropic metals cf. Ref. 10, from which expressions analogous to (1) and (2) can readily be obtained for the anisotropic case.

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Investigation of Nuclear Energy Levels by Measuring the Angle of Emission of the Reaction Products

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SIMPLE considerations based on the laws of the conservation of energy and momentum show that for the investigation of the energy levels of atomic nuclei it is possible to use a method permitting us to obtain information on the excited nuclear states by studying the angular distribution of the fissioning nuclei at a fixed direction of emission of the lighter particle, and not by measuring the energy of the light component of the reaction products, as it is generally done. This method was used for the study of the excited states of some light nuclei. The simultaneous registration of both particles was achieved with the aid of a coincidence system. Of the large number of problems associated with the investigation of the possibilities of the method under consideration, in the present paper we will discuss only its experimental realization and testing by studying the known reactions Be^9 (d, p) Be^{10} and Be^9 (d, H^4) Li^7

A beam of deuterons having an energy of 4.04 mev, obtained in the Moscow State University Cyclotron, after passing a system of diaphrams, was introduced on to the target. The transverse cross section of the beam on the target had the dimensions 2 × 3 mm. The current intensity in the beam was 5×10^{-8} A, the quantity of electricity per Faraday cylinder was registered by an integrator. The variation in energy did not exceed 40 kev. The protons and the α-particles, emitted as a result of reactions at the angle θ_2 to the direction of the incident beam, were registered by a proportional counter. The fissioning nuclei emitted from the target at an angle θ_3 to the direction of the incident beam were registered by an Allen type electron multiplier.[1] In order to avoid intensive overloading of the multiplier with foreign particles of low energy and with hard quanta of ultraviolet radiation, thin organic films were placed in front of the multiplier. The pulses of the proportional counter and of the electron multiplier were fed to the system of coincidences with a resolving time $\tau = 1 \times 10^{-6}$ sec. To attain a gradual change of the position of the counter and of the multiplier (angles θ_2 and θ_3) without distorting the vacuum, a "reaction" chamber was constructed with a mobile vacuum compressor of a special design.[2] The targets were prepared by evaporating (in a vacuum) powdery beryllium on organic films and dissolving the films in acetone. In order to avoid any considerable multiple scattering of the fissioning nuclei in the targets, the latter must by very thin and smooth. The thickness of the target in the experiments described here was $30 \pm 10\ \mu g/\ cm.$[2] To obtain the required smoothness of the target, a procedure such as proposed in the paper by Il'chenko[3] was used.

The results of the investigation of the angular distribution of the fissioning nuclei in the reaction Be^9 (d, p) Be^{10} at $\theta = 35°$ are given in Fig. 1, where conventional units are plotted along the ordinate. The diaphram in front of the counter had the dimensions 4 × 4 mm, and that in front of the multiplier 20 × 2 mm. In order to remove the peaks corresponding to the elastic scattering of deuterons in the reactions (d, p) on the nuclei O^{16} and C^{12} (contamination), an aluminum foil was placed in front of the counter (for the exclusion of deuterons), and the total thickness of the organic film in front of the multiplier was selected such that the nuclei O^{17} and C^{13} would be absorbed. Use of the laws of the conservation of energy and momentum permits us to draw the conclusion that the peak at $\theta = 74°$ corresponds to the ground state of the Be^{10} nucleus and at $\theta = 52°$ to the first excited state (3.37 mev).[4] This second peak was found to be fairly broad, which is caused mainly by the γ-radiation of the fissioning nuclei and to a lesser degree by the multiple scattering of the fissioning nuclei in the target.

The angular distribution of the fissioning nuclei in the reaction Be^9 (d, He^4) Li^7 at $\theta_2 = 50°$ is shown in Fig. 2. The arrows with the notation Li^7, Li^{7*} and Li^{7**} give the position of the peaks corresponding to the ground and to the first two excited states of the Li^7 nucleus (0.48 and 4.61 mev) assumed in accordance with calculations. The obtained peaks relate to the ground and the first excited (0.48 mev) levels of the Li^7 nucleus. No peak corresponding to the level 4.61 mev[5] was detected with any noticeable intensity, which represents a direct substantiation of the fact that the Li^7 nucleus, having been produced in this state, disintegrates into the He^4 and H^3 nuclei. The above fact illustrates that the method of "angular emission" may be useful for the study of the relative intensities of γ-radiation and of the dissociation of excited nuclei.

In conclusion I consider it a pleasant duty to

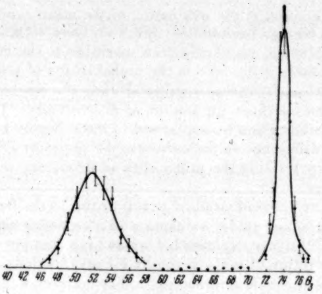

FIG. I.

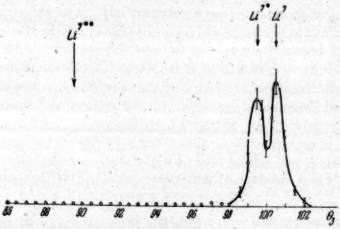

FIG. 2.

express my gratitude to S. S. Vasil'ev for his attention to this work and for a number of valuable comments, also to the engineers V. S. Zazulin, A. I. Akishin, Iu. A. Vorob'ev, A. N. Boiarkina and to the co-workers maintaining the cyclotron for their help in carrying out the various stages of this investigation.

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Mean Excitation Energy of Fissioning Uranium Nuclei on Absorption of Slow π^--Mesons

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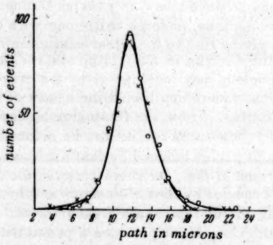
FISSIONING of uranium nuclei upon absorption of slow π^--mesons[1] may be considered as a fission induced by fast nucleons resulting from the interaction between a π^--meson and a pair of nucleons of the nucleus (n,p) or (p,p).* The fast nucleons produced may, in passing through a uranium nucleus, undergo collisions with the nucleus, giving rise to a nuclear-cascade process, and leave the nucleus in an excited state. The excited nucleus may lose energy by the evaporation of nucleons, undergoing fissioning at any of the stages of excitation. Thus, the fissioning on capture of slow π^--mesons in reality can be reduced to the fission of nuclei induced by fast nucleons, and a comaprison of the data characterizing the fission of U^{238} nuclei byslow π^--mesons and by protons of 140 mev energy represents great interest. Furthermore, the comparison provides a possibility of obtaining an evaluation for the mean excitation energy of the fissioning uranium nuclei on capture of slow π^--mesons.

In the Figure the distribution curves are given as a function of the paths of the single fragments for the fission induced by π^--mesons (crosses) and by protons with $E = 140$ mev (circles). There is good agreement between the two curves and they have one clearly expressed maximum. The coincidence of these curves, however, does not yet permit us to draw the conclusion that the mean excitation energies of the fissioning nuclei are equal in both cases, since in the region of comparatively larger mean excitation energies (from 80 to 160 mev) the shape of these curves does not change greatly.[4] It is significant that in fission by π^--mesons there is a single clearly expressed maximum, indicating that the mean excitation energy in this case is definitely higher than 50 mev. (It is known that for the energy of a falling nucleon 45 mev, the distribution curve of the single fragments for uranium has two additional clearly expressed maxima[5]).

An evaluation of the upper limit of the mean excitation energy for π^--mesons can be obtained by comparing the average number of charged evaporation particles per single fission: for slow π^--mesons[1] and for protons with $E = 140$ mev[4] these are 0.04 and 0.14, respectively; the probabilities of fission are, respectively, 0.42[1] and 0.77[6] ($Z^2 A$ = 35.0 and 34.4). If, in accordance with our evaluation[4], the mean excitation energy is equal to 80 mev in the fission by protons of 140 mev, then for uranium nuclei fissioned by π^-–mesons this value should be less than 80 mev, since in this case the average number of charged evaporation particles (0.04) is less than the corresponding value (0.14) in fission by protons of $E = 140$ mev. Therefore we may assign the following limits to the mean excitation energy in the fission of uranium on absorption of slow π^-–mesons:

$$50 \text{ mev} \ll E_{\text{exc}} < 80 \text{ mev}.$$



This value differs somewhat from those given in the literature data. Thus, in investigation[7], the mean excitation energy of heavy nuclei in emulsion (Ag, Br) on absorption of slow π^-–mesons was evaluated as ~ 100 mev. However, a conclusion may be drawn that the mean excitation energy of the Ag, Br nuclei is considerably less than 100 mev, on comparison of some available experimental data. Thus, in the interaction of protons with $E = 400$ mev with the heavy nuclei of emulsion[8,9], the average number of charged evaporation particles is equal to approximately two (1.86 and 2.1). The mean excitation energy for these nuclei, calculated by the Monte Carlo method, for the interaction of protons of this energy is equal to 50 mev[10]; however, the average number of evaporation particles on capture of π^-–mesons in the Ag, Br nuclei is considerably smaller and is equal to 1.1.[7] Even if we consider that in the interaction with protons many of the events without the ejection of ionized particles were missed, and if, accordingly, we take into account this possibility and recalculate the data for the π^-–mesons, we obtain for the upper limit of the average number of charged particles the value of 1.72. From this it follows that the mean excitation energy of the Ag, Br nuclei from π^-–mesons does not exceed 50 mev, which does nor contradict the evaluation of the mean excitation energy for uranium which we have obtained. In conclusion, we should draw attention to the considerable difference in the probabilities of uranium 0.42[1] and 0.75[6] on capture of slow π^-– mesons and on interaction with protons of $E = 140$ mev. This difference can be explained to some degree by the difference in the values of the paramter Z^2/A, characterizing the probability of fissioning in both cases.

The above-described peculiarities in the fission of uranium nuclei on capture of π^-–mesons may be qualitatively explained if we assume that on absorption of a π^-–meson by a heavy nucleus the most probable interaction is that between a meson and a pair of nucleons of the outer shell of the nucleus. Then it is most probable that only one of the produced fast nucleons will pass through the nucleus, undergoing collisions, while the other will leave the nucleus without interacting with the nucleons. From this it follows directly that the mean excitation energy of the nucleus cannot exceed 70 mev, since each of the interacted nucleons will receive an energy of ~ 70 mev.

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*From the data of the investigations,[2,3] the interaction of a π^-–meson is most probable with the (n, p) pair.

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Compton Effect in the Extended Electron

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IT is well known that the introduction of cutting-off form factors into the mathematical formalism of the present field theory results necessarily in the condition of the gauge invariance relations.[1] In this case the supplementary Lorentz condition $\partial A\mu / \partial x\mu = 0$ is no longer an integral of motion, and all four possible polarizations of the γ-quanta will contribute to the effective cross sections $d\sigma = d\sigma_{\perp} + d\sigma_{\parallel}$. However, if the contribution of the γ-quanta with longitudinal and scalar polarizations was small, i.e., if $d\sigma_{\parallel} << d\sigma_{\perp}$, then the theory could be considered as a first approximation of a more exact theory which would satisfy exactly the conditions for gauge invariance, and in which $d\sigma_{\parallel}$ would vanish. If, in the opposite case, the effective cross section $d\sigma_{\parallel}$ is comparable to $d\sigma_{\perp}$, then the possibility of emitting scalar and longitudinal γ-quanta is a basic result of the theory with a form factor. Let us consider these problems in the simple case of the Compton scattering.

The matrix element of the Compton effect on an extended electron will be written in the form:

$$<p_2 s_2;\ k_2 e_2 |S| p_1 s_1;\ k_1 e_1>$$
$$= \frac{ie^2}{(2\pi)^6} \frac{1}{2\sqrt{k_1 k_2}} \int d^4 (x_1 \ldots x_3') F(x_1 x_2 x_3) F(x_1' x_2' x_3')$$
$$\times \bar{u}(p_2) \gamma_\mu \bar{S}(x_3 - x_1') \gamma_\nu u(p_1) e^{-ip_2 x_1 + ip_1 x_3'}$$
$$\times \left\{ e^{ik_1 x_3' - ik_2 x_2} e_1^\nu e_2^\mu + e^{ik_1 x_2 - ik_2 x_2'} e_1^\mu e_2^\nu \right\}.$$

Here p_1, s_1 and p_2, s_2 are the energy-momentum and spin vectors of the electron in the initial and final states: k_1, e_1 and k_2, e_2 are the energy-momentum and polarization vectors of the γ-quanta in the initial and final states. The theory of non-local interactions not being gauge invariant, one has to take into account all four possible and independent polarizations $\mathbf{e}/\mu$ of the γ-quantum $\mathbf{k}_1$, and all four possible and independent polarizations $\mathbf{e}_{2\mu}$ of the γ-quantum $\mathbf{k}_2$. If ξ_μ is the fraction of the γ-quanta with a polarization $\mathbf{e}_{1\mu}$ in the scattered beam, then

$$d\sigma(p_2;\ k_2 e_2 \mid p_1;\ k_1)$$
$$= \sum_{\mu=1}^{4} \xi_\mu d\sigma(p_2;\ k_2 e_2 \mid p_1;\ k_1 e_{1\mu}) \Big/ \sum_{\nu=1}^{4} \xi_\nu .$$

In the particular case where all four independent polarizations have equal probabilities, i.e., when $\xi_1 = \xi_2 = \xi_3 = \xi_4$:

$$d\sigma(p_2;\ k_2 \mid p_1;\ k_1) \tag{1}$$
$$= \frac{r_0^2 r_2^2 d\Omega}{4k_1^2} \left\{ \frac{1}{k_1} \Phi_1^2 \left[k_2 + m\left(1 + \frac{m}{k_1}\right)\right] \right.$$
$$+ \frac{1}{k_2} \Phi_2^2 \left[k_1 - m\left(1 - \frac{m}{k_2}\right)\right]$$
$$\left. + m\left(\frac{1}{k_1} - \frac{1}{k_2} - 2\frac{m}{k_1 k_2}\right)\Phi_1 \Phi_2 \right\},$$

$$\Phi_1 = \varphi(p_2;\ p_2 + k_2)\, \varphi(p_1 + k_1;\ p_1); \tag{2}$$
$$\Phi_2 = \varphi(p_2;\ p_2 - k_1)\, \varphi(p_2 - k_1;\ p_2); \quad \varphi(p;\ q)$$
$$= (2\pi)^8 \int F(x_1 x_2 x_3)\, e^{-ip(x_1 - x_2) - iq(x_2 - x_3)}\, d^4(x_1 x_2 x_3);$$

where θ is the angle between the vectors $\overline{\mathbf{k}_1}$ and $\overline{\mathbf{k}_2}$.

If the energy k_1 of the scattered γ-quantum and the angle θ are such that $k_1 (1 - \cos\theta) >> m$ then $k_1 >> k_2$ and $\phi_2 \approx 1$ for $\lambda \lesssim 10^{-12}$ cm. (It follows from the comparison with the experimental data that $\lambda < 1.2 \times 10^{-13}$; see below.) In this case

$$(d\sigma / d\sigma_{\text{loc}})(p_2;\ k_2 \mid p_1;\ k_1) \approx 1/2. \tag{3}$$

The right-hand side of (3) differs from ½ only in the case of small angles (for $k_1 \gtrsim 10^9$ ev; $\theta \lesssim 3$ or 5°)

$$(d\sigma / d\sigma_{\text{loc}})(p_2;\ k_2 | p_1;\ k_1) = \Phi_1^2 \to 0;\ \vartheta \sim 0;\ k_1 \to \infty. \tag{4}$$

As the energy of the scattered γ-quantum increases, this region becomes smaller.

The effective cross section (1) accounts for the transverse, as well as for the longitudinal and scalar polarizations of the γ-quantum. It is interesting to consider also the effective cross section $d\sigma_{\perp}(p_2;\ k_2 / p_1;\ k_1)$ for the case where the polarization of the γ-quanta is purely transverse.

If it happens that for $\xi_2 = \xi_4 = 0$ the difference $d\sigma_{\parallel} = d\sigma - d\sigma_{\perp} << d\sigma_{\perp}$, then the probability of emission of longitudinal and scalar γ-quanta, in the theory with non-local interaction, is negligibly small. We get by a standard method:

$$
\begin{aligned}
& d\sigma_{\perp}(p_2; k_2 \mid p_1; k_1) \qquad (5) \\
& = \frac{r_0^2 k_2^2}{2k_1^2 m} d\Omega \left\{ \frac{1}{k_1} \Phi_1^2 (mk_2 + k_1^2 \sin^2 \vartheta) \right. \\
& \quad + \frac{1}{k_2} \Phi_2^2 \left(mk_1 - \frac{1}{2} k_2^2 \sin^2 \vartheta \right) \\
& \quad \left. + \Phi_1 \Phi_2 \left(\frac{1}{2} k_2 - k_1 - m \right) \sin^2 \vartheta \right\},
\end{aligned}
$$

where $\phi_i = \phi_i(\theta)$ as informula (2).

The cross section (5) for Compton scattering differs considerably from the effective cross section got by Klein, Nishina and Tamm for energies $k_1 \sim \hbar \lambda / c$ and for small scattering angles. Comparing with (1), it follows that the γ-quanta with longitudinal and scalar polarizations contribute considerably to the cross section. As the energy k_1 of the scattered γ-quanta increases, the magnitude of the effective cross sections σ and $\sigma_{\perp}$ decreases rapidly.

Lawson[3] has measured the Compton scattering effective cross section for an energy $k_1 = 80$ mev of the γ-quanta. The experimental error is of 15% and the result agrees with the cross section computed by the Klein-Nishina-Tamm formula. It follows from these experiments that in any case $\lambda < 10^{-12}$ cm. More exact measurments are reported[4] for an energy $k_1 = 250$ mev; the experimental error is of 10%, and the result also agrres with the Klein-Nishina-Tamm calculation. It follows from Eqs. (1) and (5) that for $k_1 = 250$ mev and $\theta = 4^\circ$ the magnitude of the constant λ cannot exceed 1.2×10^{-13} cm. This result is obtained for form-functions $\phi_1 = e^{-2\lambda^2 m k_1}$, $\phi_2 = e^{-2\lambda^2 m k_2}$, but it does not change appreciably when the form of these functions is varied.

To conclude, let us note that the study of the creation of electron-positron pairs by cosmic-rays electrons with energies of 0.1 to 10 bev, in photo-emulsions, also yields results in agreement with the calculations performed by the known methods of the theory of local interaction.[5] For electrons with energies of 100 bev there are some indications of disagreement with theory. However, the large experimental errors prohibit the drawing of any definite conculsions.

I express my gratittude to Prof. D. I. Blokhintsev for his interest and for his valuable suggestions.

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Relation between Neutron Scattering in Polycrystals and Specific Heat

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THE cross section for the inelastic coherent scattering of thermal neutrons in polycrystals, relative to a single nucleus and for an arbitrary frequency spectrum, is expressed in the following way, correct to terms of order $1/M$ (that is, neglecting multiple phonon processes and thermal factors):*

$$
\sigma_{\pm}(E, \omega)\, d\omega = \frac{\sigma_0}{3M} \sqrt{1 \mp \hbar\omega/E}\, (2 \mp \hbar\omega/E) \qquad (1)
$$

$$
\times \frac{(\pm 1)}{1 - e^{\mp \hbar\omega/T}} \frac{E}{\hbar\omega} \nu(\omega)\, d\omega .
$$

Here $\sigma_{\pm}(E, \omega)\, d\omega$ is the scattering cross section for a neutron with energy E, as a result of which a phonon with a frequency in the interval $d\omega$ is excited (or absorbed); σ_0 is the cross section for the scattering of a neutron by a single massive nucleus, M is the mass number of a nucleus of the crystal, T is the temperature (in units of energy and $\nu(\omega)$ is the frequency spectrum of the crystal, relative to a single nucleus.

The crystalline frequency spectrum entering into Eq. (1) is in turn linked in a well-defined manner with the lattice specific heat at constant volume, $C(T)$, by the relation

$$C(T)=\frac{d}{dT}\int_0^\infty\frac{\hbar\omega\nu(\omega)}{e^{\hbar\omega/T}-1}\,d\omega\,, \tag{2}$$

and in principle can be determined from experimental values of the specific heat. Such a procedure is subject to errors of considerable magnitude which are difficult to control, however,**[2] particularly in the high-frequency region; consequently, calculations of the differential cross sections σ_+ from experimental specific heat data can scarcely lead to dependable results.

It is possible to do considerably better with those characterestics of neutron scattering which depend on integrals over the frequency spectrum, such as, for example, the total cross section for inelastic scattering

$$\sigma=\int_0^{E/\hbar}\sigma_+(E,\,\omega)\,d\omega+\int_0^\infty\sigma_-(E,\omega)\,d\omega\equiv\sigma_++\sigma_-, \tag{3}$$

the moments of the energy which can be transferred

$$\langle\Delta E^k\rangle=\frac{1}{\sigma}\Big[\int_0^{E/\hbar}\sigma_+(E,\omega)(\hbar\omega)^k d\omega \tag{4}$$

$$+(-1)^k\int_0^\infty\sigma_-(E,\omega)(\hbar\omega)^k d\omega\Big]$$

and so forth, since these quantities are relatively insensitive to the details of the frequency spectrum. Furthermore, quantities in which the integral is taken over the entire frequency spectrum can be immediately expressed in terms of integrals over the specific heat along the real axis, so that it is possible to determine the error in the calculation directly. We will demonstrate this procedure in calculating the quantity σ_-, which for very cold neutrons coincides with the total coherent cross section, since the elastic cross section and σ_+ are equal to zero.[1]

First of all we note that from specific heat data it is possible to calculate the quantity***

$$\int_0^T C(t)\,dt=\int_0^\infty\frac{\hbar\omega\nu(\omega)}{e^{\hbar\omega/T}-1}\,d\omega \tag{5}$$

$$=\sum_{n=1}^\infty\int_0^\infty\hbar\omega\nu(\omega)\,e^{-n\hbar\omega/T}\,d\omega=\sum_{n=1}^\infty\psi\left(\frac{T}{n}\right)$$

$$=\sum_{n=1}^\infty\sum_{k=1}^\infty\mu_k\int_0^{T/kn}C(t)\,dt$$

$$=\sum_{q=1}^\infty\int_0^{T/q}C(t)dt\sum_{k:q}\mu_k=\int_0^T C(t)\,dt$$

$$\psi(T)=\int_0^\infty\hbar\omega e^{-\hbar\omega/T}\nu(\omega)d\omega=\sum_{k=1}^\infty\mu_k\int_0^{T/k}C(t)\,dt\,,$$

where the μ_k are the familiar coefficients appearing in the series expansion of the Dirichlet function $1/\xi(x)$[($\xi(x)$ is the Riemann ζ-function)], which, as is well known, are equal to 1 for $k=1$, to $(-1)^l$ if k is the product of l with different prime numbers, and are zero otherwise. Consequently, for any function $\nu(\omega)$ which falls to zero faster than ω, the integral over which converges, we have identically (for $p < 2$):

$$\int_0^\infty\frac{\nu(\omega)d\omega}{\hbar\omega}e^{-\hbar\omega/T}\frac{(\hbar\omega)^p}{\sqrt{E+\hbar\omega}} \tag{6}$$

$$=\frac{e^{E/T}}{\Gamma(^1/_2)}\int_0^T\frac{ds}{s^2}\left(\frac{1}{s}-\frac{1}{T}\right)^{-1/2}e^{-E/s}\Big\{\frac{1}{\Gamma(2-p)}$$

$$\times\int_0^s\left(\frac{1}{t}-\frac{1}{s}\right)^{1-p}\psi(t)\,\frac{dt}{t^2}\Big\}\,.$$

For $p = 2$ it is necessary to replace the integral in the curly brackets by $\psi(s)$, and for $p > 2$ by $d^{p-2}\psi(s)/d(1/s)^{p-2}$. Therefore, it is possible to obtain an expression for σ_- immediately, in view of the fact that

$$\sigma_-=\sum_{k-1}^\infty\frac{\sigma_0}{3M}\int_0^\infty\frac{E}{\hbar\omega}\,\frac{1}{\sqrt{1+\hbar\omega/E}} \tag{7}$$

$$\times\left(2+3\,\frac{\hbar\omega}{E}+\frac{\hbar^2\omega^2}{E^2}\right)e^{-n\hbar\omega/T}\,\nu(\omega)\,d\omega\,.$$

In particular, for small energies,

$$\sigma_-=\frac{2}{3\sqrt{\pi}}\,\frac{\sigma_0}{M}\sqrt{\frac{T}{E}}\int_0^1\frac{\sqrt{1-x}}{\sqrt{x}}\,dx \tag{8}$$

$$\times\sum_{q=1}^\infty\frac{1}{\sqrt{q}}\,C\left(\frac{T}{q}\,x\right)\sum_{k/q}\frac{\mu_k}{\sqrt{k}}\,.$$

In the Figure we present graphically the temperature dependence of σ_- for graphite, for neutrons with energies of 17.2° K, calculated from specific heat data[4] using the exact Eqs. (6) and (7) (solid curve), and from the approximate formula (8) (broken curve). Experimental values are shown for comparison.[5] The error in these calculations is < 2–3%. A curve calculated on the basis of the Debye approximation is also shown[5] (dot–dash line). As can be seen from these curves, the exact theoretical curve satisfactorily describes the temperature behavior of the experimental values of σ_-, but differs from them by an amount which indicates the presence of an additional temperature–independent scattering cross section of

~ 0.2 barn. This discrepancy can be linked with incoherent scattering by impurities in graphite, or with multiple scattering through small angles.

In an analogous manner it is possible to derive expressions for other integrals over the frequency spectrum, for example for σ_+ for large energies, when the first integral in (3) can be extended out to infinity, etc. For instance, the average value of the cross section $\bar{\sigma}_{\text{eff}} = \overline{\sigma v}/\bar{v}$ and the moments of the energy transferred to (or from) neutrons with Maxwellian distribution are:

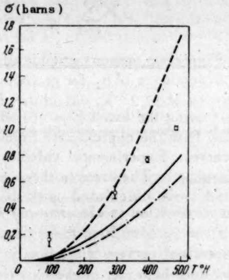
$$\bar{\sigma}_{\text{eff}} = \frac{1}{T^2}\int_0^\infty \sigma(E)\,Ee^{-E/T}\,dE,\quad \langle\overline{\Delta E^k}\rangle \tag{9}$$

$$= \frac{1}{\bar{\sigma}_{\text{eff}}\,T^2}\int_0^\infty \sigma(E) < \Delta E^k > Ee^{-E/T}\,dE,$$

which likewise can be described in relation to the specific heat. Thus, after substituting (3) into (9) and making simple transformations of the integrals we obtain for $\bar{\sigma}_{\text{eff}}$:

$$\bar{\sigma}_{\text{eff}} = \frac{\sigma_0}{3\,MT^2}\left\{\int_0^1 \sqrt{1-x}\,(2-x)\,\frac{dx}{x^4} \times\sum_{k=0}^{\infty}\chi\left[\frac{1}{T}\left(\frac{1}{x}+k\right)\right] + \int_0^\infty \sqrt{1+x}\,(2+x)\,\frac{dx}{x^4}\sum_{k=1}^{\infty}\chi\left[\frac{1}{T}\left(\frac{1}{x}+k\right)\right]\right\}; \tag{10}$$

$$\chi\left(\frac{1}{t}\right) = -\frac{d}{d(1/t)}\,\psi(t).$$



In conclusion I express my profound gratitude to V. L. Ginzburg for valuable discussions while this work was being carried out.

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*These formulas appear in the elementary generalization of the corresponding expression for Debye polycrystals (see Ref. 1, for example), when it is possible to replace the sum over the vectors of the reciprocal lattice by an integration; this may always be done for σ_-, but, for σ_+, only when the neutron wavelength is much less than the dimensions of the elementary cell of the crystal.

**This difficulty is connected with the fact that the transformation which is the inverse of the transformation of the form $g(p) = \int_0^\infty f(x)\,(e^{px}-1)^{-1}\,dx$ requires knowledge of the function $g(p)$ in the complex plane of the variable p.

***This possibility results from the fact that the sum of μ_k over all divisors of any number q ($\sum_{k/q}\mu_k$) is equal to unity for $q = 1$ and to zero for $q \neq 1$. Consequently, we then have

$$\int_0^T C(t)\,dt = \int_0^\infty \frac{\hbar\omega\nu(\omega)}{e^{\hbar\omega/T}-1}\,d\omega$$

$$= \sum_{n=1}^{\infty}\int_0^\infty \hbar\omega\nu(\omega)\,e^{-n\hbar\omega/T}\,d\omega = \sum_{n=1}^{\infty}\psi\left(\frac{T}{n}\right)$$

$$= \sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\mu_k\int_0^{T/kn} C(t)\,dt$$

$$= \sum_{q=1}^{\infty}\int_0^{T/q} C(t)\,dt\sum_{k/q}\mu_k = \int_0^T C(t)\,dt.$$

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Angular Distribution of the Uranium Fission Fragments Produced by High-Energy Neutrons

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MANY investigations made in the past few years have shown that the fragments of heavy nuclei fissioned by either charged or neutral medium energy (up to 20 mev) particles display a parallel anisotropy relative to the direction of the bombarding beam.[1,2] It was also established that this orientation becomes perpendicular[3,4] for uranium at incident-proton energies of 460 mev and higher. This communication reports on the results of analogous experiments with fission of uranium by high-energy neutrons.

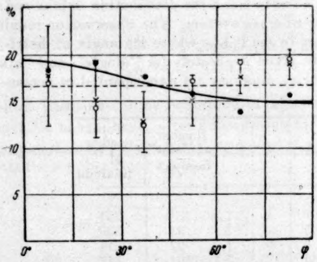
Plates coated with fine-grain nuclear emulsion P–9 impregnated with uranium salt were exposed to a collimated beam of neutrons obtained by charge-exchange between 680 mev protons and a beryllium target. The plane of the emulsion was parallel to the neutron beam. The plates were exposed behind a concrete wall in a supplementary holder made of cadmium and boron. The neutron sensitivity of the emulsion, judged from the number of π-μ decay events, was approximately 25–30 mev.

The angles between the fission fragments and the projection of the beam on the plane of the field of view were measured. Whenever the fragments subtended an angle less than 180°, the angle was read from the line joining the ends of their traces, so as to make the distribution approximate a center-of-mass system. The observation results are given in the Table, where the angle of distribution is given separately for "single" and "star" cases, (i.e., fissions not accompanied or accompanied by emission of one or several charged particles). The next to the last line of the Table gives the anisotropy coefficient determined as the ratio of the number of events occuring at 45–90° angles to the number of events with angles in the 0–45° interval; the statistical errors are also indicated. Even though the latter are indeed great, one is immediately struck by the fact that this coefficient systematically exceeds unity for all groups of "star" fissions.

Number of rays / Angles of projection, deg	0	1	2	3	4	5–6	Number of "star" fissions	Total number of fissions
0—15	128	69	20	9	4	1	103	231
15—30	127	60	23	7	—	3	93	220
30—45	119	52	28	5	6	1	92	211
45—60	111	61	25	11	2	2	101	212
60—75	105	66	26	15	4	3	114	219
75—90	121	81	29	9	3	2	124	245
Bcero	711	389	151	56	19	12	627	1338
Anisotropy coefficient	0.90	1.15	1.13	1.50			1.18	1.03
Statistical errors . . .	±0.07	±0.12	±0.20	±0.32			±0.10	±0.06

A method described in Ref. 5 was used to convert the observed angular distribution in a plane to a three-dimensional one. The Diagram shows the resulting angular dependence of the fragments of single and "star" fissions. For comparison, the Diagram shows the results of observations of fission of uranium by 460 and 660 mev protons taken from Refs. 3 and 4 and transformed by the above-mentioned method to obtain the three-dimensional picture. Although approximately the same number of fragments due to fission by high-energy neutrons escapes at 0° as at 90°, there is an obvious anisotropy in their distribution, similar to that obtained by fission with fast protons, where the anisotropy coefficient is 1.27 ± 0.05. The lower perpendicular directivity of neutron-induced fission can be fully explained by the complex energy make-up of the bombarding beam. The distribution of single fissions is in good agreement with the relationship obtained in Ref. 1, for uranium fission with neutrons up to 14 mev.

Based on data of the ray distribution of the fission events produced by 140–660 mev protons,[6] and assuming that the distribution obtained for fissions by neutrons of the same energy would be similar, one can obtain the general distribution from the number of rays produced in the fission of uranium by a neutron beam having a spectrum of the form given in Ref. 7, using at the same time the known relationship for $\sigma_f(E)$.[8] The distribution so computed agrees with that observed. A considerable contribution to the number of single fissions is made here by relatively low-energy neutrons which indeed determine the character of the angular anisotropy. Increasing the energy of the incident neutrons increases the proportion of events corresponding to a higher excitation of the fissioning nucleus, and the parallel anisotropy is replaced by a perpendicular one, in the sense defined above.



Angular distribution of fragments relative to the direction of the incident particle at a 15° angle interval and per unit solid angle: O—proton-induced fission, per Refs. 3 and 4, ●—neutron-induced "single" fissions, ×—neutron-induced "star" fission. The distribution of fragments as per Ref. 1 (solid curve) and the isotropic distribution (dotted line) are shown. The statistical errors are indicated for star fissions. The points corresponding to the proton experiments have errors that are approximately half as large.

We also determined the anisotropy in the angular distribution of the particles that accompany the fission of uranium nuclei, and compared them with the corresponding experimental data on proton-induced fission by protons.[6] The observed directivity of the particles produced in single-ray fissions is high—the forward to backward ratio is 2.0 ± 0.2; this ratio becomes 1.7 ± 0.2 for 2-ray fissions and 1.3 ± 0.2 for multiple-ray fissions. Calculations analogous to those made for the ray distribution, show that these quantities would result from experiments with a proton beam of a similar spectral composition.

One must thus assume that the anisotropy in the escape of fragments, the distribution as obtained from the number of particles accompanying the fragments, and the directivity of these particles are approximately the same whether the uranium nuclei are fissioned by high-energy neutrons or by protons of the same energy.

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On Molecular Neutronoscopy

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THE use of high-power short monoenergetic neutron pulses makes it possible now to devise a new method, one that can be called "molecular neutronoscopy," for research on the structure and properties of molecules.

The idea in this method is to subject the molecules under investigation to bombardment by short "packets" of monoenergetic neutrons ($E_0 \approx 1$–10 ev) and to determine the binding energy of the molecules, the probability of various molecular conversions induced by the neutrons, and certain other characteristics (mentioned below) from the type of the time-of-flight spectrum and of the angular distribution of the scattered neutrons. The method proposed resembles most closely in its potentialities and means of realization the investigation of Debye levels of crystal lattices with the aid of "cold" neutrons.[1,2]

When the molecules, bombarded by neutrons of energy E_0, do not break up into two "chips" (atom, radical, or ion) the spectrum of the neutrons scattered at an angle θ consist of a series of lines E_{ni}, whereby

$$\sqrt{E_{ni}} = (A+1)^{-1}[\sqrt{E_0}\cos\theta + \sqrt{E_0(A^2-\sin^2\theta)-Q_i A(A+1)}], \tag{1}$$

where Q_i are the energies of the excited levels and A is the mass number of the molecules. However, if $E_0 > E_b(A+1)/A$, it becomes possible to have inelastic scattering with molecule break up and the spectrum of the scattered neutrons will contain groups with energies between $E_{n\,\max}$, as given by (1) (putting $Q_i = E_b$), and $E_{n\,\min}$, which vanishes when $E_0 > E_b A/(A-1)$ and given by

$$\sqrt{E_{n\,\min}} = (A+1)^{-1}[\sqrt{E_0}\cos\theta - \sqrt{E_0(A^2-\sin^2\theta)-E_{\mathrm{CB}} A(A+1)}] \tag{2}$$

at

$$E_b(A+1)/A < E_0 < E_b A/(A-1).$$

Analysis of the position and intensity of the lines and line bands in the neutron spectrum relative to the time of flight $t_i = 72l/\sqrt{E_{ni}}$ microseconds (where l is the distance from the scatterer to the detector in meters and the energy is in ev) affords many opportunities for determining the energy of the bonds broken by the neutrons, the probabilities of such breaks, and the excitation levels of the molecules.

Further possibilities of the method proposed are related to the development of theoretical concepts of the scattering of neutrons with excitation of molecules (see, for example, Refs. 3 and 4 for the case of scattering with molecule breakup).

Considerable information on the rotational and oscillation states and on the corresponding wave functions of the molecules is obtained from analysis of the data on the total cross section and angular distribution of the scattering neutrons, and also from the spectrum of the neutrons scattered with molecule breakup at a given angle (typical probelm of pulse approximation).

Let us estimate the practical realizability of the method of molecular neutronoscopy. At the present time it is actually possible to irradiate areas up to 5 cm^2 (for example, 0.7 × 7 cm) in mechanical monochromators by means of a flux of monoenergetic resonant neutrons on the order of 10^5–10^6 cm^{-2} sec^{-1}. By using as a second scattering or absorbing filter a 5–10 cm diameter rotor with groups of through slots approximately 0.5 millimeters wide, placed 60° apart, it is possible to obtain a 2-microsecond neutron pulse every 100-200 microseconds, provided the linear speed of the rotor edges reaches 250 meters per second (still approximately half the now attainable value). A scatterer of sufficient thickness will scatter in this case $5 \times 10^3 - 10^5$ neutrons per second. A detector comprising ten counters, filled (to approximately 1 atmos), with $B^{10}F_3$, 2 cm in diameter and 50 cm long (with approximate efficiency of 10%), located one meter from the scatterer, will count from 8.5×10^4 to 1.7×10^6 neutrons during an experiment lasting six hours.

Let us estimate the accuracy of the experiments, using as an actual example the irradiation of water (in the gas phase, with the H–OH binding energy being approximately 5.18 ev) with 7 ev neutrons, the neutron scattering being observed at an angle of 150°. Here the maximum energy of the neutrons scattered upon breakup of the H_2O molecules is approximately 1.1 ev and at $l = 1$ meter we have correspondingly $t_{\min} = 70$ microseconds. The maximum time of flight, determined by cutting off the secondary neutrons having $E_n \lesssim 0.3$ ev with a cadmium filter, is approximately 130 microseconds. The relative energy spread of the primary neutrons at the flux value indicated above is approximately 10%. This error—the greatest of all— decreases in proportion to the primary-neutron flux. The width of the neutron pulse is approximately 3% of the minimum time of flight of the scattered neutrons. The effective thickness of the scatterer, equal to the range of the primary neutrons in water, is approximately 0.6 cm, and results in a time-of-flight spread of 2-2.5% (allowing also for the counter diameter, which equals 2 cm). The spread in the velocity of the scattered neutrons due to thermal motion of the molecules in the scatterer is approximately $\sqrt{kT/AE_n}$ and is close to 3.8%. The errors due to the spread in the counter discharge time (approximately 0.1 microseconds) and the angular resolution ($\Delta\theta \approx 1.5°$) are much smaller and can be disregarded. As a result we obtain for this case $\Delta E_b / E_b \approx 13.7\%$, with almost the entire total error (approximately 97%) being due to the spread in the energy of the primary neutrons. Reducing the primary neutron flux and $\Delta E_0 / E_0$ by 10 times decreases the overall error in the determination of the binding energy to 2.7%.

Molecular neutronoscopy cannot compete with optical methods and with radio spectroscopy whenever the latter methods (which are more accurate) can be used to investigate a substance without a change in its aggregate or chemical state (which may be accompanied by a change in the very properties that are being studied).

However, molecular neutronoscopy does have the advantage of being capable of investigating molecules in specimens in any state, and is consequently capable of studying intermolecular interaction, i.e., the influence of such factors as temperature, pressure, aggregate state, presence of outside admixtures, etc., on the binding energy, and perhaps also other molecular properties mentioned above. In addition, molecular neutronoscopy uncovers still another specific possibility of interest to radiation chemistry and radiation biology, namely, a means of establishing the relative and absolute probability of breaking various molecular bonds by neutron excitation of molecules at different levels.

In conclusion I express my gratitude to F. L. Shapiro and M. I. Pevzner, whose discussions contributed much to establishing the possibility of the method proposed here.

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Excitation Function for the $Si^{28}(d, p)Si^{29}$ Reaction

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USING a method described in Ref. 1, we investigated the yields of various groups of protons from the Si^{28} (d,p) Si^{29} reaction, as functions of the fixed detector angle and the primary deuteron energy. In a stripping reaction, the ratios of these yields, called hereinafter "differential excitation functions" for short, should depend in a characteristic manner on the moment of momentum introduced by the neutron in the formation of a finite nucleus in a definite excited state.

However, the stripping mechanism is not unique to the (d, p) reaction and in some cases a substantial contribution to the cross section of the reaction is introduced by the mechanism of the intermediate nucleus. The latter mechanism is of resonant character, corresponding to the formation of excited levels of the intermediate nucleus. The cross section of the (d,p) reaction consists in this case of a contribution from the stripping mechanism, a contribution from the intermediate-nucleus mechanism, and a contribution corresponding to the interference between the stripping mechanisms and the compound nucleus. The presence of such resonances in the differential excitation functions and their interference character were recently established experimentally for light nuclei.[2-9]

We obtained spectra of protons from the Si^{28} (d,p) Si^{29} reaction at an angle $\theta = 109°$ with the direction of the motion of the primary particles for 15 values of deuteron energy ranging from 1.75 to 4.75 mev. A typical Si^{28} (d,p) Si^{29} proton spectrum at $E = 3.45$ mev is shown in Fig. 1. The proton groups $p_0, p_1, p_2, p_3, p_4, p_5$ correspond to excited levels of Si^{29} at $E_{exc} = 0, 1.28, 2.03, 2.43, 3.07$, and 3.62 mev, respectively. Figure 2 shows the ratios of the differential excitation functions of various levels of the final Si^{29} nucleus obtained from these spectra.

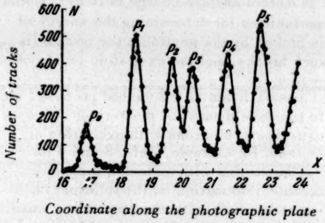

FIG. 1. Distribution of the tracks on a photographic plate. N—number of tracks, X—coordinate along the photographic plate.

These relationships disclose 3 resonances at incident-deuteron energies $E_d = 3.26$, 3.75 and

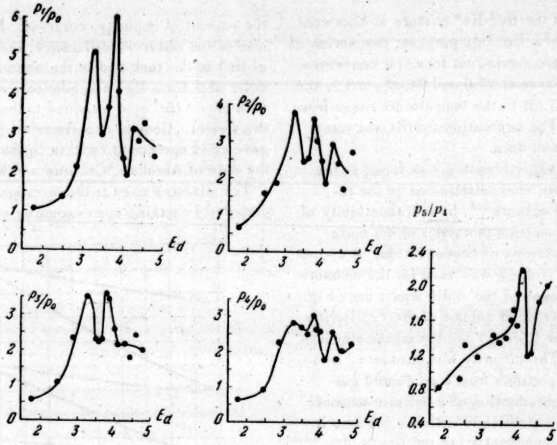

FIG. 2.

4.23 mev, corresponding to the formation of new excited (14.8, 15.3 and 15.7 mev) levels in the P^{30} nucleus. It was observed that these resonances vary with the levels of the final nucleus. The first two (E_{exc} = 1.48 and 15.3 mev) are most clearly pronounced for the ratio p_1 / p_0, and their intensities diminish with diminishing kinetic energy of the proton (i.e., as the ratios p_2/p_0, p_3/p_0, etc., are reached). The third resonance (E_{exc} = 15.7 mev) is most clearly pronounced for the ratios p_5/p_0 or p_5/p_4. Inasmuch as $k_pR \geq 2.5$ (where k_p is the proton wave vector and R the nucleus radius), it is impossible to determine uniquely the spins and parities of the state of the intermediate nucleus from the known spins and parities of the final nucleus.

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Measurement of the Saturated Vapor Pressure of a He^3-He^4 Mixture with a High He^3 Concentration

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IN connection with the separation of He^3 from He^4 by rectification, it was necessary to know the phase diagram of the He^3-He^4 mixture at high concentrations of He^3. For this purpose, two series of measurements were carried out for molar concentrations of He^3 in vapor at 97.0 and 94.0%, and in the liquid at about 93.6% in the temperature range from 1.45 to 2.0° K. The separation coefficient was computed from these data.

The saturated vapor pressure was found in the vapor from a known concentration as in the researches of other authors[2,3], by a discontinuity of the pressure composition isotherm, which corresponds to the beginning of condensation. A copper vessel of 31 cm^3 volume was used for the measurement. The thickenss of the walls was 1 mm (Fig. 1*a*). The effective dead volume of the capillaries (internal diameter 1.5 mm) and the manometer was about 150 cm^3. The impurity was introduced into the apparatus in portions from a graduated gas tank. After the introduction of a definite amount, the pressure was established within 0.5-1 mm; it did not change subsequently (in the limits of accuracy of measurement, ± 0 2 mm Hg). The mean error in the determination of the saturated vapor pressure, i.e., the point of onset of condensation, was ± 0.5 mm.

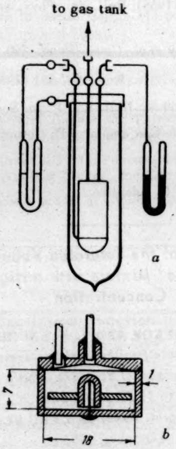

FIG. 1

Control experiments with pure He^4 showed that the isotherms, taken for increase and decrease of the amount of impurity, coincide. For the calculation of the observed difference, the vapor pressure of He^4 in the tank and in the apparatus (of the order of 0.1 mm Hg) a condensation thermometer with liquid He^4 was attached to the lower part of the vessel. Control experiments with pure He^3 gave good agreement (within limits of error) with the data of Abraham, Osborne and Weinstock[4].

The mixtures used in these experiments were prepared by mixing corresponding quantities of pure He^3 and He^4; the concentration $X = N^3/(N^3 + N^4)$ was determined with an accuracy to within ±0.1%.

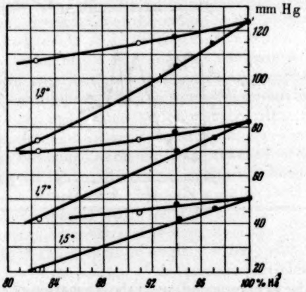

FIG. 2. O — according to Ref. 3; ● — according to the present research.

For the second series of experiments, we used a copper container of diameter 18 mm and height 7 mm (Fig. 1*b*). Slow condensation of the mixture with a concentration of 94.0% He^3 filled the vessel with liquid to about 80-90% of its volume. Before each measurement, the pressure was applied for three minutes at the given temperature. This procedure was sufficient to guarantee that the pressure was established (to within the accuracy of the experiment, ±0.2 mm Hg). A mixer was provided in the apparatus in the form of a magnetized rod that could be set into motion by switching electromagnets. However, it appeared to be unnecessary, since the mixing after the three minute exposure did not change the pressure that had been established. The measurements were taken through the temperature interval from above and below; no hysteresis was observed. The precise value of the concentration in the liquid was determined by the quantity of the mixture in the apparatus, the densities of the liquid and the vapor, and the concentration in the vapor, which could be determined from

TABLE

in vapor	T°K	P, mm Hg	in liquid	T°K	P, mm Hg
$X_v = 97\%$ He^3	1.497	46	$X_l = 93.6\%$ He^3	1.464	42.4
	1.652	66.7		1.483	45.7
	1.953	127		1.571	57.2
	2.048	151.5		1.657	70
				1.794	94.3
$X_v = 94\%$ He^3	1.509	42		1.861	108.2
	1.648	61		1.915	119.7
	1.963	115		1.975	135.2
	2.048	140			

the pressure on the basis of the results of the first set of experiments. From the relatively weak dependence of the vapor pressure on the concentration in the liquid in the concentration region studied (1 mm at 0.5% He^3 for 2° K), the values of the pressure, obtained from experiments with several different amounts of liquid and, consequently, different concentrations (from 93.8% to 93.4% He^3), lay at the limits of accuracy of the pressure measurement on one curve. For this same reason it was possible to neglect the change in the vapor concentration by diffusion (during the course of the experiment) in the conducting tubes.

The temperature of the apparatus was determined from the vapor pressure of He^4 in the tank (taking into account the corrections mentioned above for the first set of experiments) to within ±0.001° K (the 1949 tables were used).

The results that we obtained are shown in the Table. Phase diagrams were constructed from these for the mixtures of He^3-He^4 for the temperature and concentration region under investigation (Fig. 2). Plotted in the same graph are the data of Esel'son and Beresniak[3]. As is evident from the graphs, our results and those of Ref. 3 are in satisfactory agreement.

The separation coefficient

$$k = X_v(1 - X_l)/(1 - X_v)X_l$$

for the values of concentration studied is equal to 3.0 (±20%) and (with the accuracy pointed out) does not depend on the temperature in the interval 1.5-2.0° K.

The authors take this opportunity to express their thanks to A. I. Filimonov for his help in carrying out the experiments.

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Ionizational Slowing Down of High-Energy Electron Positron Pairs

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THE components of a high energy electron-positron pair of energy E separate with the small angle $\theta \sim mc^2/E$. The interference of the electron and positron fields at a small distance from the point of creation leads to a smaller ionization than that which would be caused by two electrons. This phenomenon was theoretically examined by Chudakov.[1] The aim of the present note is to give another derivation of the equations of the ionizational slowing down of a pair and to clarify the limitations of such a development.

We make use of a method first shown by Landau (Ref. 2, p. 145) for the derivation of the equations of ionization loss at high energies (in the region of the polarization effect). If the positron and electron are at a particular moment of time t at points $\mathbf{r}_1(t)$ and $\mathbf{r}_2(t)$ then T, the energy loss of the pair in a unit of time, is

$$T = ec\,\{\mathbf{v}_1 \mathbf{E}(\mathbf{r}_1, t) - \mathbf{v}_2 \mathbf{E}(\mathbf{r}_2, t)\}, \tag{1}$$

where $c\mathbf{v}_1$ and $c\mathbf{v}_2$ are the speeds of the positron and electron, and E is the electric field of the pair. The last can be considered as a field in the macroscopic realm caused by charge densities ρ and currents j,

$$\rho = e\delta(\mathbf{r} - \mathbf{r}_1(t)) - e\delta(\mathbf{r} - \mathbf{r}_2(t));$$
$$\mathbf{j} = ec\,\{\mathbf{v}_1 \delta(\mathbf{r} - \mathbf{r}_1(t)) - \mathbf{v}_2\delta(\mathbf{r} - \mathbf{r}_2(t))\}$$

and can be presented in the form of the following Fourier integral

$$\mathbf{E}(\mathbf{r}, t) = (ie/2\pi^2)\int d\mathbf{k}\,\{e^{i\mathbf{k}(\mathbf{r}-\mathbf{r}_1)}[\mathbf{v}_1(\mathbf{k}\mathbf{v}_1) - \mathbf{k}/\varepsilon(\mathbf{k}\mathbf{v}_1)]/[k^2 - (\mathbf{k}\mathbf{v}_1)^2\varepsilon(\mathbf{k}\mathbf{v}_1)] - e^{i\mathbf{k}(\mathbf{r}-\mathbf{r}_2)}[\mathbf{v}_1(\mathbf{k}\mathbf{v}_2) - \mathbf{k}/\varepsilon(\mathbf{k}\mathbf{v}_2)]/[\mathbf{k}^2 - (\mathbf{k}\mathbf{v}_2)^2\varepsilon(\mathbf{k}\mathbf{v}_2)]\}, \tag{2}$$

where $\epsilon(\omega)$ is the dielectric constant of the medium corresponding to a frequency ω. In the derivation of the (2) the trajectories of the particle being scattered in the medium can be considered as straight line segments for regions in which slowing down is still important.

Substituting Eq. (2) into Eq. (1) we obtain $T = 2T_0 - T_1$, where T_0 is the ionizational slowing down of a single electron, and T_1 is the interference term

$$T_1 = \frac{ice^2}{2\pi^2}\int d\mathbf{k}\left\{\frac{(\mathbf{v}_1\mathbf{v}_2)(\mathbf{v}_1\mathbf{k}) - (\mathbf{k}\mathbf{v}_2)/\varepsilon(\mathbf{k}\mathbf{v}_1)}{\mathbf{k}^2 - (\mathbf{k}\mathbf{v}_1)^2\varepsilon(\mathbf{k}\mathbf{v}_1)}e^{i\mathbf{k}(\mathbf{r}_2-\mathbf{r}_1)} + \frac{(\mathbf{v}_1\mathbf{v}_2)(\mathbf{v}_2\mathbf{k}) - (\mathbf{k}\mathbf{v}_1)/\varepsilon(\mathbf{k}\mathbf{v}_2)}{k^2 - (\mathbf{k}\mathbf{v}_2)^2\varepsilon(\mathbf{k}\mathbf{v}_2)}e^{-i\mathbf{k}(\mathbf{r}_2-\mathbf{r}_1)}\right\}.$$

In the calculation of T_1 it is important that the transverse component of the pair separation be larger than the parallel component. Indeed the last is proportional to $v_1 - v_2 \sim (mc^2/E)^2$ while the first is determined by the angle of separation of the pair $\theta \sim mc^2/E$ and by the multiple scattering angle. Therefore, having selected the Z axis along the direction of $\mathbf{v}_1$ or $\mathbf{v}_2$, we can substitute $k_x s$ for $\mathbf{k}(\mathbf{r}_2 - \mathbf{r}_1)$ in the exponent, where $s = (x_2 - x_1)$. After this the k_z integration can be carried out in the same way as done by Landau in the calculation of T_0. It turns out that the limiting expression for ϵ for high frequencies is of importance in the integral, $\epsilon = 1 - \lambda^2 c^2/\omega^2$ where $\lambda^2 = 4\pi n c^2/mc^2$, and n is the number of electrons in the unit volume. We obtain

$$T_1 = \frac{ce^2\lambda^2}{\pi}\int\frac{\cos k_x s}{k_x^2 + k_y^2 + \lambda^2}\,dk_x\,dk_y = 2e^2c\lambda^2 K_0(s\lambda), \tag{3}$$

where k_0 is the corresponding cylindrical function.

The convergence of this integral is shown by the fact that the interference effects depend on large distances for which the macroscopic viewpoint is valid. The analogous integral for T_0, as is known, diverges and must be limited by some maximum value of the transverse wave vector K_n which is related to the energy E_n transmitted to the atomic electron.

For large s ($s\lambda >> 1$) the interference term disappears, as can be seen from Eq. (3). For small s ($s\lambda << 1$), one can use the relationship $k_0(z) = \ln(2/\gamma z)$, where $\gamma = e^C = 1.781$. Then

$$T_1 = 2e^2c\lambda^2 \ln(r_{\max}/s),$$

where $r_{\max} = 2/\gamma\lambda$.

If T_0 is written in analogous form (see, for example, Ref. 2)

$$T_0 = ce^2\lambda^2 \ln(r_{\max}/r_{\min}),\quad r_{\min} = a(\hbar/mc)\sqrt{mc^2/E_m}$$

($a = 1.85$), then T can be written in the following form obtained in Ref. 1.

$$T = 2T_0 \ln(s/r_{\min})/\ln(r_{\max}/r_{\min}). \tag{4}$$

Entering into $r_{\min}$ is the quantity E_m which is the maximum energy transmitted to an atomic electron as determined from experimental data. We would like to express our thanks to L. D. Landau for discussion of the results.

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Diffraction Scattering of High-Energy Photons by Nuclei

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THE properties of the nucleus with a respect to photons of high energy (for $kR \ll 1$, where k is the wave number of the photon, R is the atomic radius) can be characterized by a complex index of refraction $n + i\varkappa/k$, where n is approximately 1 and $\varkappa R \ll 1$. The magnitude of the absorption coefficient $\varkappa$ can be expressed in terms of general formulas in terms of the experimentally determined cross section for photo-meson production on nuclei:

$$\varkappa R = 3\sigma_c^2 / 4\pi R^2. \tag{1}$$

The existence of absorption must lead to strong scattering of photons. Using general diffraction relations for polarized nuclei[1] it is easy to show that the cross section for scattering σ_s is

$$\sigma_s = 9\sigma_c^2 / (32\pi R^2). \tag{2}$$

The scattering amplitude at small angles θ is

$$f(\theta) = ik\varkappa \int_0^R J_0(k\theta\sqrt{R^2 - s^2})\, s^2\, ds,$$

from which we find for the differential cross section

$$\begin{aligned} d\sigma_s / do &= {}^1\!/_2\, \sigma_s (kR)^2 \Phi^2 (kR\theta), \\ \Phi(x) &= x^{-2}(x^{-1}\sin x - \cos x). \end{aligned} \tag{3}$$

In agreement with the experimental data[2] at photon energies of the order of 300 mev, σ_c is approximately $10^{-28} A$ cm^2. In this case the scattering cross section must be

$$\sigma_s = 10^{-30}\ \text{см}^2 \text{ for Be}, \quad \sigma_s = 0{,}9 \cdot 10^{-28}\ \text{см}^2 \text{ for U}.$$

Let us compare the diffraction scattering with scattering of photons by a Coulomb field. The cross section of the last σ_γ for $E \gg mc^2$ is equal[3] to

$$\sigma_\gamma = 8{,}5 \cdot 10^{-35} Z^4 \text{ cm.}^2$$

Thus the ratio σ_s / σ_γ changes from 50 for Be to 10^{-2} for U, that is for heavy nuclei the diffraction scattering is considerably smaller than the coherent scattering by the charge. Nevertheless, it must appear as a consequence of a different angular distribution. In agreement with Eq. (3), diffraction scattering is effective at an angle $\theta_s \sim 1/kR$ while scattering by the Coulomb field is concentrated in the region $\theta_\gamma \sim mc^2/E$. Therefore, for $E = 300$ mev, the differential cross sections for U are comparable for $\theta = 0.015$, after which $d\sigma_\gamma / d\theta$ rapidly decreases, while $d\sigma_s / d\theta$ remains in this region at a constant value which is equal to 0.8 mb ($\theta_s = 0.09$).

We would like to express appreciation to K. A. Ter-Martirosian for discussing this problem.

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Concerning the Impulse Approximation

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BRUECKNER[1] has examined the problem of the scattering of a particle by a system of two scatterers with zero-range forces [the scattering from each of these centers is spherically symmetric and is characterized by the amplitude $\eta = (1/k) \times \sin\delta\, e^{i\delta}$, where δ is the phase of the S-wave at infinity]. For the scattered amplitude in this problem, we obtain the following expression:

$$\begin{aligned} f(\vartheta) = \left(1 - \eta^2 \frac{e^{2ikR}}{R^2}\right)^{-1} \Big[\eta \big(e^{i(\mathbf{k}_0 - \mathbf{k})\mathbf{r}_A} + e^{i(\mathbf{k}_0 - \mathbf{k})\mathbf{r}_B}\big) \\ + \eta^2 \frac{e^{ikR}}{k^2} \big(e^{i(\mathbf{k}_0\mathbf{r}_A - \mathbf{k}\mathbf{r}_B)} + e^{i(\mathbf{k}_0\mathbf{r}_B - \mathbf{k}\mathbf{r}_A)}\big)\Big], \end{aligned} \tag{1}$$

where $\mathbf{k}_0$ and $\mathbf{k}$ are the wave vectors before and after scattering, $\mathbf{r}_A$ and $\mathbf{r}_B$ are the radius vectors of the scattering centers, and $R = |\mathbf{r}_A - \mathbf{r}_B|$.

From this expression, Brueckner, using a well-known theorem relating the imaginary part of the scattering amplitude in the forward direction with the total cross section, obtains the latter. Comparing this expression for the total cross section with the corresponding one obtained with the aid of the impulse approximation, the author shows that the difference between these two expressions becomes insignificant not for $R \sim \infty$, but for $\delta \to 0$ (for simplicity, it is assumed that the amplitude η is the same for both scatterers). From this the conclusion is reached that the use of the impulse approximation without taking account of multiple scattering is valid only when the Born approximation is applicable.

In reality, however, this conclusion is true only for the total cross section (and even then with reservations, which will be discussed below). Calculation of the differential cross section for small angle scattering on the basis of the impulse approximation gives correct results, which, evidently, is physically related to the fact that for small angle scattering interference of the wave scattered by each of the centers becomes significant, and this is correctly taken account of in the impulse approximation. Indeed, from Eq. (1) we obtain the following exact expression for the differential cross section $d\sigma/d\Omega$ per unit solid angle, averaged over all directions of the vector R:

$$\frac{d\sigma}{d\Omega}=2\frac{d\sigma_0}{d\Omega}\,\frac{1+\dfrac{\sin(|\mathbf{k}_0-\mathbf{k}|R}{|\mathbf{k}_0-\mathbf{k}|R}+\dfrac{\sin^2\delta}{\lambda^2}\left(1+\dfrac{\sin|\mathbf{k}_0+\mathbf{k}|R)}{|\mathbf{k}_0+\mathbf{k}|R}\right)+4\sin\delta\cos(x+\delta)\dfrac{\sin x}{\lambda^2}}{(1-x^{-2}\sin^2\delta)^2+4x^{-2}\sin^2\delta\sin(x+\delta)}, \tag{2}$$

where $x = kR$, and $d\sigma_0/d\Omega = k^{-2}\sin^2\delta$ is the differential cross section for scattering by one of the centers.

In the impulse approximation we obtain, with no difficulty, the expression

$$d\sigma/d\Omega = 2(d\sigma_0/d\Omega)\{1+\sin(|\mathbf{k}_0-\mathbf{k}|R)/|\mathbf{k}_0-\mathbf{k}|R\}. \tag{3}$$

For large incident energies ($kR \gg 1$) and small scattering angles ($\theta \lesssim 1/kR$) Eqs. (2) and (3) differ only by small quantities of the order of x^{-2}. Therefore, for these conditions ($\eta/R \ll 1$), the impulse approximation, as could have been expected leads to the correct results, which are identical with the exact ones for $kR \to \infty$. For large scattering angles, however, the second term in the curly brackets of Eq. (3) (whose absolute value is of the order of $1/x$) oscillates rapidly*, and therefore its contribution to the total cross section is small, of the order of x^{-2}. This explains why the expression obtained for the total cross section in the impulse approximation differs, in this case, from the exact one** [compare Eqs. (6) and (5) of Brueckner[1]] by a quantity of the same order of magnitude as those retained in the impulse approximation.

Let us note, in addition, that for $kR \gg 1$, both the exact formula and that obtained in the impulse approximation lead to a result according to which the total cross section is, to a high degree of accuracy, equal to the sum of the cross sections for each of the centers [there is a deviation only for terms of the order of $(kR)^{-2}$].

I should like to thank Professor K. A. Brueckner for discussions concerning this problem during the Moscow conference of May, 1956.

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* Thus, Eqs. (2) and (3) differ by quantities which are small in comparison with those retained in the impulse approximation for all values of ϑ except for small intervals in the neighborhood of the zeros of the function $\sin(2x\sin\vartheta/2)$.

** It is not difficult to show that integrating expressions (2) and (3) over all scattering angles leads to results identical with those for the total cross sections obtained by Brueckner[1].

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Scattering of Fast Neutrons by a Nuclear Coulomb Field

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As is known, the principal contribution to the scattering cross section of neutrons scattered by nuclei is made by the nuclear forces. Other effects, due to the interaction between the magnetic and perhaps also the electric moments of the neutron and the nuclear Coulomb field are also to be expected. The interaction of the neutron magnetic moment with the nuclear Coulomb field was theoretically investigated by Schwinger[1] and Sample[2]. Hereafter, we shall call the scattering that results from this interaction the Schwinger scattering. The Schwinger scattering cross section is practically independent of the energy.

The question of the existence of an electric dipole moment in the neutron was already discussed in the literature[3]. It follows from Smith's experiments[4] that the electric dipole, if it exists, should be less than or approximately equal to $5 \times 10^{-21} e$ CGS, where e is the electronic charge. In addition to intrinsic electric dipole moment, the neutron may exhibit (in a strong Coulomb field) an electric moment caused by the deformation of the meson shell. The neutron becomes "polarized", so to speak. This problem has not been investigated theoretically or experimentally.

The above phenomenon will contribute the most to the scattering cross section if the neutrons are scattered by heavy nuclei, and should manifest itself in an anomalous behavior of the differential scattering cross section at small angles, since the Coulomb forces act at greater than nuclear distances. A simple estimate shows that the effect due to an intrinsic neutron electric dipole moment of the order of magnitude indicated above is negligibly small.

Attempts by Longley and others[5] to detect Schwinger scattering of neutrons by lead gave inconclusive results. An analogous attempt by Sample and others[6] was also unsuccessful.

The angular distribution of fast neutrons emerging from a reactor and scattered by Pb and Cu was studied and the data given below are the preliminary results of the investigation. The neutron beam was restricted by a steel collimator to 0.9 × 3.6 cm.

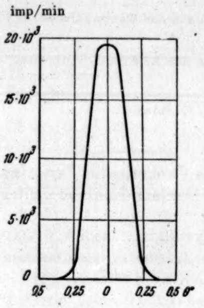

FIG. 1

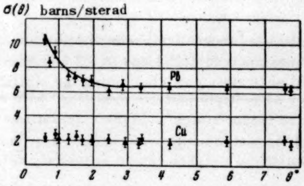

FIG. 2

Scattering was effected with a Pb or Cu plate 1 cm thick mounted 10 cm from the edge of the collimator. The detector, located 325 cm from the plate, was a photomultiplier with plastic scintillator (ZnS in plexiglass), having a low sensitivity to gamma-rays and to neutrons with energies above 1.5 mev. The degree of collimation is characterized by the curve of Fig. 1. Before starting the work, the total effective number of neutrons incident on the scatterer was determined, so as to permit subsequent calculation of the differential scattering cross section $\sigma(\theta)$. The effective neutron energy as determined from the nuclear scattering cross section, was 3-4 mev.

The measured angular distribution is shown in Fig. 2, the curve being plotted from Schwinger's theoretical equation [Eq. (10) of Ref. 1]. The measurement results show the increase in cross-section, characteristic for the Schwinger scattering for Pb ($Z = 82$) at angles less than 2°. The value of the cross section is in agreement with the Schwinger and Sample theoretical investigations. For Cu ($Z = 29$) the increase in cross section is within the limits of experimental error.

In a recently published work, Voss and Wilson[7] report observed Schwinger scattering of 100 mev neutrons by uranium. The variation of the scattering cross section with the angle is close to the theoretical curve, but the authors did not cite the numerical values of $\sigma(\theta)$.

In conclusion, one must remark that estimates indicate that if the neutron exhibits a "polarizability" $\alpha \sim r^3$ (where r is the nucleon dimension), the additional contribution to the cross section of the scattering of neutrons by heavy nuclei becomes noticeable. The effect increases with diminishing energy, but more careful experiments are needed for its detection.

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Multiple Formation of Particles in 5.3 bev Nucleon-Nucleon Collisions

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We calculated theoretically the distribution of nucleon-nucleon collisions at 5.3 bev from the number of secondary particles, using the statistical theory of multiple-particle formation[1] with and without the isobar states[2]. In the calculations we employed the method suggested in Ref. 3, with which statistical weights can be accurately calculated.

The percentage statistical weights of the various processes are given in Table I. A classification by charged state, as required for conservation of the isotopic spin, is given in Table II for (p-p)-collisions and in Table III for (n-p)-collisions (N---nucleon, N'---isobar state, M---number of pions). Thus, for example, for (p-p)-collisions the process $NN\ 2\pi$ (the statistical weight of which is indicated in Table I) gives a probability of 0.300 for the charged state ($pp+-$), a probability of 0.100 for the charge state ($pp00$), etc. (see Table II).

From the data cited it is easy to obtain the distribution of the inelastic collisions from the number of charged particles ("prongs") which, in the case of (p-p)-collisions, can be compared with the experimental data by Fowler and others[4]. Such a comparison is shown in Table IV. It is seen from this Table that allowing for the resonant interaction between the nucleons and mesons by introducing the isobar states leads to a better agreement with experiment.

In conclusion, I thank I. L. Rozental' for useful advice.

We note with gratitude the constant interest of the late Professor S. Z. Belen'kii, who stimulated the performance of the calculations.

TABLE I

Number of mesons	Type of process	Statistical Weight (%)		Number of mesons	Type of process	Statistical Weight (%)	
		$p-p$	$n-p$			$p-p$	$n-p$
0	NN	0.3	0.4	3	$NN3\pi$	4.5	4.5
1	$NN\pi$	6.5	6.8		$NN'2\pi$	31.8	31.0
	NN'	1.0	0.7		$N'N'\pi$	11.7	11.1
2	$NN2\pi$	11.5	12.0	4	$NN'4\pi$	2.7	2.7
	$NN'\pi$	16.7	17.4		$NN'3\pi$	1.2	1.2
	$N'N'$	0.9	1.2		$N'N'2\pi$	11.2	11.1

TABLE II

Number of Mesons	Charged State	Probabilities of Charged States of Various Processes		
		$NNm\pi$	$NN'(m-1)\pi$	$N'N'(m-1)\pi$
0	*pp*	1.000		
1	*pp* 0	0.250	0.167	
	pn +	0.750	0.833	
2	*pp* + −	0.300	0.350	0.200
	pp 00	0.100	0.117	0.178
	pn + 0	0.450	0.483	0.578
	nn ++	0.150	0.050	0.044
3	*pp* + − 0	0.267	0.280	0.244
	pn ++−	0.333	0.360	0.422
	pp 000	0.033	0.033	0.030
	pn + 00	0.233	0.247	0.252
	nn ++ 0	0.134	0.080	0.052
4	*pp* ++−−	0.122	0.131	0.119
	pp +− 00	0.180	0.190	0.186
	pn ++−0	0.408	0.431	0.480
	nn +++−	0.082	0.060	0.036
	pp 0000	0.012	0.013	0.014
	pn + 000	0.106	0.110	0.112
	nn ++ 00	0.090	0.065	0.053

TABLE III

Number of Mesons m	Charged State	Probabilities of Charged States of Various Processes		
		$NNm\pi$	$NN'(m-1)\pi$	$N'N'(m-2)\pi$
0	*pn*	1.000		
1	*pp* −	0.278	0.167	
	pn 0	0.444	0.666	
	nn +	0.278	0.167	
2	*pp* − 0	0.189	0.137	0.067
	pn + −	0.466	0.563	0.733
	pn 00	0.156	0.163	0.133
	nn +−	0.189	0.137	0.067
3	*pp* +−−	0.138	0.124	0.076
	pp − 00	0.100	0.087	0.078
	pn +− 0	0.462	0.508	0.611
	nn ++−	0.138	0.124	0.076
	pn 000	0.062	0.070	0.081
	nn + 00	0.100	0.087	0.078
4	*pp* +−− 0	0.179	0.163	0.133
	pn ++−−	0.209	0.229	0.267
	pp − 000	0.048	0.043	0.035
	pn +− 00	0.316	0.338	0.380
	nn ++− 0	0.179	0.163	0.133
	pn 0000	0.021	0.021	0.017
	nn + 000	0.048	0.043	0.035

TABLE IV

Number of "prongs" (inelastic interactions)	Number of Events		
	Experiment of Ref. 4	Theoretical (isobar states included)	Theoretical (isobar states not included)
2	14	15.1	21.5
4	16	16.3	10.5
6	2	0.6	0.5

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Consequences of the Renormalizability of Quantum Electrodynamics and Meson Theory

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THE consequences of the renormalizability of quantum electrodynamics and meson theory which have been obtained by Gell-Mann and Low[1] and Bogoliubov[2] are most easily formulated, in our opinion, in the following way. We shall start from the following equations of Gell-Mann and Low[1]

$$\alpha(g_0^2, \xi - L) = \frac{\alpha_c(g_c^2, \xi)}{\alpha_c(g_c^2, L)}, \tag{1}$$

$$\beta(g_0^2, \xi - L) = \frac{\beta_c(g_c^2, \xi)}{\beta_c(g_c^2, L)}, \quad d(g_0^2, \xi - L) = \frac{d_c(g_c^2, \xi)}{d_c(g_c^2, L)}.$$

Here α_C, β_C and d_C are the asymptotic expressions for the slowly-varying factors of the renormalized vertex parts and Green's functions for the nucleon and meson*, g_C is the renormalized meson coupling constant, $\xi = \ln(-k^2/m^2)$, $L = \ln(\Lambda^2/m^2)$ (Λ is the momentum "cutoff"). The quantities α, β, d, g_0 are the nonrenormalized quantities corresponding to the cutoff momentum.

For convenience, we have introduced the logarithmic variables ξ and L from the beginning. In addition to the trivial inference that α, β and d become unity for $\xi = L$, Eq. (1) includes the statement, fundamental in what follows, that for $\xi >> 1$, α, β and d asymptotically approach functions only if the difference $\xi - L = \ln(-k^2/\Lambda^2)$, i.e., no longer depend on the nucleon mass m.

We then introduce a quantity which may be called "the effective coupling constant"

$$g^2(\xi) = g_0^2 \alpha^2(g_0^2, \xi - L)\, \beta^2(g_0^2, \xi - L)\, d(g_0^2, \xi - L) \tag{2}$$

$$= g_c^2 \alpha_c^2(g_c^2, \xi)\, \beta_c^2(g_c^3, \xi)\, d_c(g_c^2, \xi).$$

The second of Eqs. (2) is obtained from (1) and from the relation between the renormalized and nonrenormalized coupling constants. From Eq. (2) it is seen that the effective coupling constant g may be considered either a function of g_0^2 and $\xi - L$, or of g_C^2 and ξ.

The final formulation consists of the assertion that the logarithmic derivatives of α and α_C, etc., with respect to ξ, which are equal according to Eq. (1), depend on one variable, namely, on the effective coupling constant

$$\alpha'/\alpha = \alpha_c'/\alpha_c = F_1(g^2); \quad \beta'/\beta = \beta_c'/\beta_c = F_2(g^2); \tag{3}$$

$$d'/d = d_c'/d_c = F_3(g^2);$$

$$(g^2)'/g^2 = 2F_1(g^2) + 2F_2(g^2) + F_3(g^2)$$

The primes here denote differentiation with respect to the arguments $\xi - L$ or ξ, whichever is appropriate. The last of Eqs. (3) follows from the first three and Eq. (2). As an example, let us prove the first of the equations. According to Eq. (2), ξ

$= \xi(g_C^2, g^2)$ and therefore the quantity α_C'/α_C, which depends on g_C^2 and ξ, can be written as a function of g_C^2 and g^2. Therefore, the ratio α'/α, which is equal to α_C'/α_C, can be written in the following form:

$$\alpha'(g_0^2, \xi - L)/\alpha(g_0^2, \xi - L) = F_1[g_c^2, g^2(g_0^2, \xi - L)].$$

We emphasize here that g^2 is considered a function of g_0^2 and $\xi - L$. If g_0^2 is held fixed, and ξ and L varied so that $\xi - L$ remains constant, then g_C^2, which depends on g_0^2 and Λ, will vary while the left side of the equation, as well as g^2 on the right side, will remain constant. This is possible only if the function $F_1(g_C^2, g^2)$ does not depend directly on g_C^2. Thus we arrive at the first of Eqs. (2). The rest of Eqs. (2) are obtained similarly.

The functions F_1, F_2, F_3 of Eq. (3) can be determined by considering values of ξ close to L. Then $\ln(\Lambda^2/-k^2) = L - \xi$ is small, and if $g_0^2 \ll 1$, then $g_0^2(L - \xi)$ is also small, i.e., usual perturbation theory is applicable, and in the second order for the symmetric pseudoscalar theory this leads to the following results (all the calculations are carried out to logarithmic accuracy; i.e., we consider only the largest logarithmically divergent part of the integrals):

$$\alpha = 1 - (g_0^2/4\pi)(\xi - L); \quad \beta = 1 + (3g_0^2/8\pi)(\xi - L);$$
$$d = 1 + (g_0^2/\pi)(\xi - L).$$

From this we obtain for $\xi \to L$

$$\alpha'/\alpha = -g_0^2/4\pi, \quad \beta'/\beta = 3g_0^2/8\pi, \quad d'/d = g_0^2/\pi.$$

From Eq. (2) for $\xi \to L$, $g^2 \to g_0^2$, and to first order in g^2, we obtain

$$F_1(g^2) = -g^2/4\pi; \; F_2(g^2) = 3g^2/8\pi; \; F_3(g^2) = g^2/\pi;$$
$$(g^2)'/g^2 = 5g^2/4\pi. \tag{3'}$$

Integrating this last equation between the limits ξ and L, and remembering that $g^2(L) = g_0^2$, we have

$$g^2(\xi) = g_0^2/Q; \quad Q = 1 + (5g_0^2/4\pi)(L - \xi). \tag{4}$$

From Eqs. (3), (3') and (4), we obtain the asymptotic expressions for the vertex parts and Green's functions of the nucleon and meson[3,4]

$$\alpha = Q^{1/5}; \quad \beta = Q^{-3/10}; \quad d = Q^{-4/5}. \tag{5}$$

Finally, we shall show for the simpler case of quantum electrodynamics how, if we know the perturbation theory series for the photon Green's function

$$d_t = \sum_{m \geq n} c_{mn} e_0^{2m} (\xi - L)^n, \tag{6}$$

we can obtain the asymptotic expression for d_t to arbitrary order in e^2. On the basis of Ward's theorem, Eqs. (2) and (3) can, for quantum electrodynamics, be written in the form

$$\begin{gathered} e^2(\xi) = e_0^2 d_t(e_0^2, \xi - L) = e_c^2 d_{t,c}(e_c^2, \xi), \\ d_t'/d^t = d_{t,c}'/d_{t,c} = F(e^2), \quad (e^2)'/e^2 = F(e^2). \end{gathered} \tag{7}$$

Considerations similar to the above give

$$F(e^2) = \sum_{m=1}^{\infty} c_{m1} e^{2m} \Big/ \sum_{m=0}^{\infty} c_{m0} e^{2m}. \tag{7'}$$

It should be emphasized that in obtaining the series (6) one must make use of the cutoff indicated by Gell-Mann and Low[1], thus assuring that the condition $d_t(e_0^2, 0) = 1$ is satisfied.

In conclusion, I should like to express my gratitude to K. A. Ter-Martirosian for many valuable comments and to I. Ia. Pomeranchuk, V. B. Berestetskii and B. L. Ioffe for fruitful discussions.

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* As an example, we are considering the symmetric pseudoscalar meson theory with pseudoscalar coupling.

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Relation between the Angular Distributions of Particles and Their Decay Products

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IN a number of cases where it is impossible to obtain direct information about the process of formation or interactions of particles, because of their short lifetime, it is necessary to limit investigations to the secondary particles, the decay products of the initial particles (for example, the γ-quanta from the decay of the π°-meson). It is essential to know how the distributions of the initial and secondary particles are related.

Let us consider the practically important case where the speed of the secondary particles is that of light and the angular distribution of the initial particles $W(\cos\theta, \varphi)$ does not depend on the azimuthal angle φ and can be represented as a linear combination of terms of the type $W_n = \frac{1}{2}(n+1)\cos^n\theta \times$ (n—integer). The distribution of the secondary particles $F(\cos\theta)$ is then an analogous linear combination consisting of terms of the type

$$F_n(\cos\vartheta) = \frac{n+1}{2\beta^n\psi(\vartheta)} \sum_{k=0}^{k=n+1} \nu_{nk}\,\delta_k, \qquad (1)$$

where β is the speed of the initial particle,

$$1/\gamma^2 = 1-\beta^2, \quad \psi(\vartheta) = 1-\beta^2\cos^2\vartheta,$$
$$\nu_{nk} = C_n^k\left(\frac{n+1}{n-k+1}\cos^2\vartheta - 1\right)\cos^{n-k}\vartheta,$$

and the functions δ_k are connected by the recurrence relation:

$$\delta_{k+2} \qquad (2)$$
$$= \frac{(1+\beta\cos\vartheta)^2(\beta-\cos\vartheta)^{k+1} - (1-\beta\cos\vartheta)^2(-\beta-\cos\vartheta)^{k+1}}{2k\gamma^2\beta\,\psi^2(\vartheta)} - \frac{\sin^2\vartheta}{\gamma^2}\frac{k+1}{k}\delta_k.$$

Eq. (2) is valid for $k \geq 1$. For $k \leq 2$ the δ_k functions have the following form

$$\delta_0 = -1, \quad \delta_1 = \cos\vartheta/\gamma^2,$$
$$\delta_2 = \sin^2\vartheta/\gamma^2 - \psi(\vartheta)\,\mathrm{Arth}(\beta)/\beta\gamma^2,$$
$$(\mathrm{Arth}\,\beta = \sqrt{1+\beta/1-\beta}).$$

Using the above relations it can be shown that for any arbitrary even (odd) n the angular distribution $F_n(\cos\theta)$ is a polynomial consisting of even (odd) powers of cosine.

Expressions for the function $F_n(\cos\theta)$ were obtained for $n \leq 6$. In view of their cumbersome form, in the present short communication we limit ourselves to the equation for the angular distribution for $n = 4$:

$$F_4(\cos\vartheta) \qquad (3)$$
$$= \frac{5}{4\beta^4}\left[2 + \frac{99-16\beta^2}{3\gamma^2} - \frac{5}{\beta\gamma^2}(7-3\beta^2)\,\mathrm{Arth}\,\beta\right]\cos^4\vartheta$$
$$- \frac{5}{2\beta^4\gamma^2}\left[15-4\beta^2 - \frac{3}{\beta\gamma^2}(5-3\beta^2)\,\mathrm{Arth}\,\beta\right]$$
$$\times \cos^2\vartheta + \frac{5}{4\beta^4\gamma^2}\left[3-2\beta^2 - \frac{3}{\beta\gamma^2}\,\mathrm{Arth}\,\beta\right].$$

A general characteristic of the functions F_n is the very rapid variations arbitrarily close to the point $\beta = 1$. Only for $\beta \approx 1$ does the angular distribution of the initial and secondary particles become similar. With decrease in β the anisotropy of the angular distribution rapidly disappears. The higher the power of n the more clearly does this characteristic appear. Even for high values of the speed β the angular distribution of the secondary particles is still close to the isotropic and high precision of measurement is required in order to determine the angular distribution of the initial particles. This has application, for example, in the investigation of the angular distribution of π^0 mesons in the vicinity of threshold. Thus if the angular distribution of π°-mesons is proportional to $\cos^2\theta$ then the share of the isotropic portion of the angular distribution of the γ-quanta for a proton energy of 660 mev is one-half, while for an energy of 340 mev it consists already of almost 90%.

Until rather large even values of the index n the roots of the equation $F_n(\cos\theta) = \frac{1}{2}$ are included in a small interval of angles around $\theta^* = \arccos(1\sqrt{3})$. An important consequence of this characteristic of the F_n functions is that the emission of secondary particles at "the isotropic" angle θ^* depends little on the speed of the initial particles (for $n = 2$ the emission does not depend on β as has been shown in Ref. 1). This allows the magnitude of the total cross section for formation of π^0-mesons in nucleon collisions to be determined from measurements of the emission of γ-quanta only at one angle. If the distribution of the initial particles contains odd powers of the cosine then to obtain the magnitude of the total cross section it is necessary to measure the emission of secondary particles at two angles θ^* and $\pi-\theta^*$. The indicated "isotropic" properties of the angular distributions of the secondary particles considerably simplify the problem of measuring the energy dependence of the total cross section particularly in the case where the angular distribution of the initial particles differs in the investigated interval of energy.

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Behavior of Particles with Nonzero Spin in Crossed Constant and Varying Magnetic Fields

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We give an exact solution for the behavior of particles of arbitrary spin in crossed constant and varying magnetic fields[1].

The wave function of the particle in a magnetic field can be written in the form

$$i\hbar d\psi / dt = -(\mathbf{H}\hat{\mathbf{M}})\,\psi, \tag{1}$$

where ψ is the $(2J+1)$-component wave function of the particle and $\hat{\mathbf{M}}$ is the magnetic moment vector operator, proportional to the angular momentum. We shall consider the case when the external magnetic field acting on the particle is composed of a constant field H_0 (along the z axis) and a varying field which has components $H_x = H_1 \cos \omega t$ and $H_y = H_1 \sin \omega t$. In this case the wave equation (1) becomes

$$i\hbar \frac{d\psi}{dt} = -\frac{\mu}{2} H_1 [e^{i\omega t}(\hat{J}_x - i\hat{J}_y) + e^{-i\omega t}(\hat{J}_x + i\hat{J}_y)]\,\psi - \mu H_0 J_z\,\psi, \tag{2}$$

where $\hat{J}$ is the angular momentum operator and μ is the magnetic moment of the particle.

Let us transform to a reference system rotating about the original z axis at a frequency ω. The components of the wave function ψ'_m in the new reference system are related to the corresponding ones ψ_m in the original system by the expression

$$\psi_m = e^{im\omega t}\,\psi'_m \qquad (-J \leqslant m \leqslant J). \tag{3}$$

Inserting expression (3) into Eq. (2), and making use of the well-known properties of the operators $\hat{J}_x \pm i\hat{J}_y$, $\hat{J}_z$, we arrive at the following equation:

$$i\hbar d\psi' / dt = (-\mathbf{H}\hat{\mathbf{M}} + \mathbf{\Omega}\hat{\mathbf{J}})\,\psi. \tag{4}$$

The components of the magnetic field vector H which enter into this equation are the following: $H_x = H_1$, $H_y = 0$, and $H_z = H_0$; Ω is the angular velocity vector $\omega\mathbf{k}$ (where $\mathbf{k}$ is the unit vector along the z axis). The operator on the right side of Eq. (4) does not depend on time and contains the term ω J, which is the "centrifugal energy" operator, whose form corresponds to the expression for the centrifugal energy in classical mechanics. Thus, Eq. (4) may be considered a wave equation in a noninertial (rotating) system of reference. Equation (4) takes on its simplest form in the (noninertial) reference system where the z axis is chosen along the vector $-\mu\mathrm{H} + \Omega$. The projection s onto angle β between this vector and the original z axis are easy to determine and are given by

$$s = \sqrt{\omega_0^2 + \omega^2 - 2\omega\omega_0 \cos\vartheta}, \tag{5}$$

$$\omega_0 = \mu H_0 / h$$

$$\beta = \mathrm{acrsin}\,\omega_0 \sin\vartheta \Big/ \sqrt{\omega_0^2 + \omega^2 - 2\omega\omega_0\cos\vartheta}, \tag{6}$$

where $\tan\vartheta = H_1/H_0$. Clearly, the solution of Eq. (4), whose initial component $\psi_m = \delta_{mm_0}$, can be written in the form[2]

$$\psi_{m_0}(t) = \sum G_{m'm_0}\{\alpha, \beta, \gamma\}\, G_{m''m'}\{\alpha, \beta, \gamma\}\, e^{im'st}\,\psi_{m'}. \tag{7}$$

The quantities $G_{m'm}\{\alpha, \beta, \gamma\}$ entering into this equation are the matrix elements of the $(2J+1)$-dimensional irreducible representation of the three-dimensional rotation group, corresponding to rotations through the Euler angles α, β, γ (see, for instance, Ref. 2).

The transition probability between states with magnetic quantum number m_0 and m (in the laboratory, not the rotating system of reference) is given by

$$R_{m_0 m} = \Big| \sum_{m'} G_{m'm_0} \{0, \beta, \pi\} G_{mm'} \{0, \tilde{\beta}, \pi\} e^{im'st} \Big|^2. \tag{8}$$

For $J = 1/2$, this expression becomes

$$R_{^1/_2, -^1/_2} = \frac{\omega_0^2 \sin^2 \vartheta}{\omega_0^2 + \omega^2 - 2\omega\omega_0 \cos\vartheta} \times \sin^2 \frac{t}{2} (\omega_0^2 + \omega - 2\omega\omega_0 \cos \omega t)^{1/2}. \tag{9}$$

In conclusion, I express my gratitude to E. Rivin for aid in the calculations.

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Concerning the Spin of the Λ^0-Particle

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WE present several general formulas obtained according to methods described elsewhere[1,2], with relation to the problem of determining the spin Λ^0-particle from the angular distribution of its decay product.

We shall characterize the spin state of an ensemble of Λ^0-particles by giving the magnitudes of the angular momentum tensors T_ν^q (defined in Ref. 1), which makes it possible to describe an arbitrary (most general) spin state of the particles in this ensemble. The angular distribution of the decay product of the Λ^0-particles ($\Lambda^0 \rightarrow p + \pi$) is of the form

$$F(\vartheta, \varphi) = \frac{w}{2\sqrt{\pi}} \sum_{q=0,2\ldots}^{2s-1} (2q+1)^{-1/2} Q(s, q) \sum_{\nu=-q}^{q} (-1)^\nu Y_{q,-\nu}(\vartheta, \varphi) T_\nu^q, \tag{1}$$

where w is the total decay probability for the Λ^0 according to the reaction $\Lambda^0 \rightarrow p + \pi^-$, s is the spin of the Λ^0, and

$$Q(s, q) = (-1)^{s-1/2-q/2} (2s+1)^{-1/2} Z(l'sl's; {}^1/_2 q).$$

The coefficients Z are tabulated by Biedenharn[3]. It can be shown that

$$Z(l'sl's; {}^1/_2 q) = (-1)^{q/2} (2l'+1)(2s+1) W(l'sl's; {}^1/_2 q) C_{l'0l'0}^{q0}$$

does not depend on l' (equal to $s + 1/2$ or $s - 1/2$) and therefore $F(\vartheta, \varphi)$ does not depend on the parity of the Λ^0-particle. Equation (1) is written in the center-of-mass system of the Λ^0-particle, and the z axis of this system will henceforth be taken parallel to the direction of motion $\mathbf{n}_\Lambda$ of the Λ^0-particle in the laboratory system [of course, Eq. (1) is valid for arbitrary choice of the z axis].

Integrating (1) over the interval of solid angle $(\varphi, \varphi + \Delta\varphi)$, $0 \leq \vartheta \leq \pi$ we can obtain the distribution in φ. The angle φ $(0 \leq \varphi \leq 2\pi)$ can be defined as the angle between the normal $\mathbf{N}$ to the production-plane of the Λ^0-particle (more exactly, $\mathbf{N} = \mathbf{n}_0 \times \mathbf{n}_\Lambda$, where $\mathbf{n}_0$ is the unit vector in the direction of the incident particles in the production reaction) and the vector $\mathbf{n} = \mathbf{n}_p \times \mathbf{n}_\Lambda$, where $\mathbf{n}_p$ is the direction of the decay proton.

$$F(\varphi) = \frac{w}{4\pi} T_0^0 \Big\{ 1 + \sqrt{2} \sum_{m=2}^{2s-1} \times \sum_{q=m}^{2s-1} [\cos m\varphi (\operatorname{Re} t_m^q) + \sin m\varphi (\operatorname{Im} t_m^q)] Q(s, q) J_{qm} \Big\}, \tag{2}$$

$$J_{qm} = \left[\frac{(q+m)!}{(q-m)!} \frac{2q+1}{2} \right]^{1/2} \frac{m}{2} 2^{1-m/2} \frac{(q/2-1)!\,(q-m-1)!!}{(q/2+m/2)!\,(q+1)!!},$$

$$t_m^q = (2q+1)^{-1/2} T_m^q / T_0^0,$$

where m and q take on only even values. This formula differs from similar ones[4,5] in that (2) contains an explicit expression for the coefficients A_M and B_M[4,5] in terms of the initial spin state of the Λ^0.

If T^q_m is real for even values of q, i.e., if $\mathrm{Im}\, t^q_m = 0$ (as can be shown[2], this will occur, for example, when the Λ^0 is produced in a reaction in which the incident particle and target are completely polarized), then $F(\varphi)$ becomes a polynomial of $\cos m\varphi$ only. The same polynomial in $\cos \eta m$ gives the distribution over the angle η between the production-plane of the Λ^0 and its decay-plane for complex values of T^q_m [compare Ref. 5, Eq. (2)].

A similar integration of Eq. (1) gives the distribution $F(\theta)$ of the number of particles emitted per unit solid angle at an angle θ to the z axis (parallel to **n**), averaged over all azimuth angles φ (see Walker and Shephard, Ref. 6):

$$F(\theta) = \frac{w}{4\pi} T^0_0 \left\{1 + \sum_{q=2,4,\ldots}^{2s-1} Q(s,q)\, P_q(\cos\theta)\, T^q_0 / T^0_0\right\}. \tag{3}$$

Comparing (2) and (3) we that see $F(\varphi)$ [or $F(\eta)$] is given by those components of the tensor T^q_m for which $q = 2, 4, \ldots, 2s-1$ and $m = 2, 4, \ldots q$, whereas $F(\theta)$ depends on entirely different components of the tensor, namely, T^q_0. The components T^q_0 and T^q_m do not determine the spin state of the Λ^0 entirely independently, but so long as they do not take on their maximum values the distribution over η can be somewhat arbitrary (within certain limits) for fixed $F(\theta)$ [cf. Ref. 7, Eqs. (17) and (18)]. Therefore, the distributions in η and $\cos\theta$ obtained by Walker and Shephard[6] are not in contradiction, which has already been noted by Morpurgo[7] who examined the cases $s = 3/2$ and $5/2$. These distributions may (the statistics are quite poor!) indicate that the Λ^0-particles observed are not entirely polarized perpendicular to the plane of the reaction $\pi^- + p \rightarrow \Lambda^0 + \theta^0$ (compare this with the note at the end of this letter**).

The observed cases of the reactions $\pi^- + p \rightarrow \Lambda^0 + \theta^0$, $\Lambda^0 \rightarrow p + \pi^-$ are at energies of about 1 and 1.5 bev. Even if the statistics at these energies were better, we could only hope to obtain a more accurate value for the lower bound of the Λ^0 spin. We shall show that a measurement of the angular distribution of the decay products of the Λ^0 produced at the threshold of the $\pi^- + p \rightarrow \Lambda^0 + \theta^0$ reaction (about 755 mev) gives the value of the Λ^0 spin itself if we make two natural assumptions, namely, that the spin of the θ^0-particle is zero (which is, at any rate, not in contradiction with the data available) and that the interaction between the Λ^0 and the θ^0 is a short range one. The latter means that we may neglect all elements of the R-matrix $(i_\Lambda i_\theta s' l' \alpha' | R^{JE} | \tfrac{1}{2} 0 \tfrac{1}{2} l \alpha)$ with $l' > 0$ in comparison with the element for which $l' = 0$ close to the threshold* (the notation is defined elsewhere[1]; the matrix R is related to the well-known S-matrix as follows: $R = S - 1$). With these assumptions we may rewrite Eqs. (7)-(9) of Ref. 1 (the target is assumed nonpolarized). Making use of the properties of the coefficients $G_{\varkappa'}$ and G_0 and the parity conservation law, introducing the notation s instead of i_Λ and suppressing the index Λ on q, we obtain the following expression for the angular momentum tensors of the Λ^0-particle with even values of q (those with odd values of q vanish):

$$T^{\varsigma 0}_{\tau 0}(\mathbf{n}_\Lambda, p_\Lambda) \tag{4}$$

$$= A(2s+1)(2q+1)^{-1/2} Q(s,q)\, Y_{q\tau}(\vartheta_\Lambda, \pi),$$

where A is a constant proportional to the total cross section for the reaction $\pi^- + p \rightarrow \Lambda^0 + \theta^0$ with $E_\pi \sim 755$-780 mev. Let us note that the index τ refers to $\mathbf{n}_\Lambda$ as the axis of quantization, and that the angle ϑ_Λ is the angle between the direction of the Λ^0-particle and the π-meson beam.

From the expression (4) for the angular momentum tensor of the Λ^0-particle, Eqs. (2) and (3) can now be used to obtain the distribution in η and θ, which will depend only on the Λ^0 spin (and on ϑ_Λ). We shall not write these general formulas here**. It is interesting, however, to note that if we integrate them over all angles ϑ_Λ, the distribution over θ is isotropic, whereas that over φ (or η) becomes

$$I_s(\eta) = C\left\{1 + \sum_{m=2}^{2s-1} \cos m\eta \sum_{q=m}^{2s-1} [Q(s,q)\, J_{qm}]^2 (2q+1)^{-1}\right\}. \tag{5}$$

For instance,

$$I_{3/2}(\eta) \sim 1 + 1/3 \cos 2\eta.$$

If the tensors of Eq. (4) are inserted into (1), we obtain a general expression for the distribution over the angle γ

$$F(\vartheta, \varphi) \qquad (6)$$

$$= \frac{Aw}{2\sqrt{\pi}} \sum_q (2q+1)^{-1} [Q(s,q)]^2 \sum_\nu Y^*_{q,\nu}(\vartheta, \varphi)\, Y_{q,\nu}(\vartheta_\Lambda, \pi)$$

$$= Aw\,[8\pi\sqrt{\pi}]^{-1} \sum_{q=0}^{2s-1} [Q(s,q)]^2 P_q(\cos\gamma) \equiv F_s(\gamma);$$

where γ is the angle between the directions (ϑ_Λ, π) and (ϑ, φ), or, as can be shown, the angle in the rest system between the direction of motion of the decay proton and that of the incident π-meson beam. The formula $F_{3/2}(\gamma) \sim 1 + 3\cos^2\gamma$ was given first by Wolfenstein[8];

$$F_{5/2}(\gamma) \sim 1 - 2\cos^2\gamma$$

$$+ 5\cos^4\gamma \sim 1 + {}^4/_5 \cos^2\gamma + {}^1/_3 \cos 4\gamma.$$

I express my gratitude to Professor M. A. Markov, who suggested the present work.

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* The angular distribution of the Λ^0 and θ^0 for 1 bev indicates the presence[6] of at least $l' = 2$. It follows from this that the range of the $\Lambda^0 - \theta^0$ forces is about 2×10^{-13} cm and that our assumption is valid for incident π-meson energies in the laboratory system from 755 to about 780-800 mev. It is, of course, possible that the matrix element $(i_\Lambda 0 i_\Lambda 0 \alpha' | R^{JE} | \tfrac{1}{2} 0 \tfrac{1}{2} l \alpha)$ is small due to some particular property of the reaction. In that case, the angular distribution of the Λ^0 and θ^0 will be nonisotropic.

** For $s = 3/2$ we have

$$F_{3/2}(\eta) \sim 1 + 0{,}5 \sin^2\vartheta_\Lambda \cos 2\eta,$$

$$F_{3/2}(\theta) \sim 1 - (1 - 3\cos^2\vartheta_\Lambda)(5 - 3\cos^2\vartheta_\Lambda)^{-1} \cos^2\theta.$$

If the angle $\vartheta_\Lambda = 90°$ we obtain $F_{3/2}(\eta) \sim 1 + 0.5 \times \cos 2\eta$, whereas $F_{3/2}(\theta) \sim 1 - 0.2\cos^2\theta$; i.e., the probability is not increased for $\cos\theta = \pm 1$, although the distribution over η is the same as that for entirely polarized Λ^0-particles.

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